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On the consistency of $X^\top AX = B$ when B is either symmetric or skew

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In this talk, we analyze the consistency of the matrix equation

$$X^\top AX = B, \tag{1}$$

where $A \in \mathbb{C}^{m\times m}$, $X \in \mathbb{C}^{m\times n}$ (unknown), and $B \in \mathbb{C}^{n\times n}$ is either symmetric or skew-symmetric. In particular, we will first provide a necessary condition for (1) to have a solution X. Then, we will prove that this condition is also sufficient for most matrices A and an arbitrary symmetric (or skew) matrix B. To be more precise, we will first show that, in order to analyze the consistency of (1), we can restrict ourselves to the case where A and B are in *Canonical form for congruence*. We use the canonical form for congruence introduced in [3], which is a direct sum of blocks of three types. Then, we will show that the condition mentioned above is sufficient when A does not contain any blocks of some of these types with certain size.

We want to emphasize that the question on the consistency of (1), when B is symmetric (respectively, skew), is equivalent to the following problem: given a bilinear form over $\mathbb{C}^n$ (represented by the matrix A), find the maximum dimension of a subspace such that the restriction of the bilinear form to this subspace is a symmetric (resp., skew) non-degenerate bilinear form.

The results presented in this talk for B being symmetric have been published in [1], whereas the ones for B being skew-symmetric are contained in the submitted manuscript [2].

References

[1] A. Borobia, R. Canogar, F. De Terán, *On the consistency of the matrix equation $X^\top AX = B$ when B is symmetric*, Mediterr. J. Math, (2021) pp. 18-40.

[2] A. Borobia, R. Canogar, F. De Terán, *The equation $X^\top AX = B$ with B skew-symmetric: How much of a bilinear form is skew-symmetric?*, submitted (2021).

[3] R. A. Horn, V. V. Sergeichuk. *Canonical forms for complex matrix congruence and $*$-congruence*, Linear Algebra Appl., 216 (2006) pp. 1010-1032.