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The Extended Aluthge Transform

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Consider a bounded linear operator T acting on a complex separable Hilbert space $\mathcal{H}$ (finite or infinite dimensional), and let $T \equiv V\,|T|$ be the canonical polar decomposition of T (that is, V is a partial isometry and $\ker T = \ker V$). The Aluthge transform of T is the operator $\Delta(T) := |T|^{1/2}\, V\, |T|^{1/2}$. For P an arbitrary positive operator such that $VP = T$, we define the *extended Aluthge transform* of T associated with P, as follows: $\Delta_P(T) := P^{1/2} V P^{1/2}$.

First, we establish some basic properties of $\Delta_P(T)$. For instance, whenever $P \geq 0$ and $VP = T$, one automatically has

(i) $|T| \leq P$,

(ii) $\ker P \subseteq \ker\ |T|$,

(iii) P commutes with $|T|$, and

(iv) the restrictions of P and $|T|$ to the range of $|T|$ agree.

We also derive a 3×3 operator matrix representation for $\Delta(T)$ and $\Delta_P(T)$ relative to the orthogonal decomposition $\mathcal{H} = \overline{\mathrm{Ran}\ |T|} \oplus \overline{\mathrm{Ran}\ (P|_{\ker T})} \oplus \ker P$. Along the way, we prove the following Intertwining Property: $|T|^{1/2}\, \Delta_P(T) P^{1/2} = P^{1/2} \Delta(T)\, |T|^{1/2}$.

Second, we study the fixed points of the extended Aluthge transform.

Third, we consider the case when T is an idempotent, and prove an optimal result for its associated Aluthge transform and extended Aluthge transform.

Next, we discuss whether the extended Aluthge transform leaves invariant the class of complex symmetric operators.

We also study how $\Delta_P(T)$ transforms the numerical radius and numerical range of T.

Finally, as a key application, we prove that the spherical Aluthge transform of a commuting pair of operators corresponds to the extended Aluthge transform of a 2×2 operator matrix built from the pair; thus, the theory of extended Aluthge transforms yields results for spherical Aluthge transforms.