Estimates on the Cheeger constant

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Isoperimetric problems

Pisa

The workshop aims at bringing together experts and young researchers working in the field of anisotropic isoperimetric problems, with a particular focus on the following subjects:

(A) Isoperimetric problems with density;

(B) Crystals and periodic structures;

(C) Gamow liquid drop model;

(D) Isoperimetric problems in geometric structures;

(E) Spectral problems.

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The Cheeger problem

Let $\Omega \subset \mathbb{R}^N$. The *Cheeger constant* of Ω is defined as

$$h(\Omega) := \inf \left\{ \frac{P(E)}{|E|}, \, E \subset \Omega, \, |E| > 0 \right\},$$

where $P(E)$ denotes the distributional perimeter of the Borel set E, and $|E|$ the standard N-dimensional Lebesgue measure.

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Spectral applications:

(i) lower bounds to the first eigenvalue of the p-Laplacian

$$\left(\frac{h(\Omega)}{p}\right)^p \leq \lambda_p(\Omega);$$

(ii) if Ω is "good enough", $h(\Omega) = \lambda_1(\Omega)$;

(iii) estimates on the L^1 norm of the p-torsion function w_p

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$$P(E) - \kappa|E|, \qquad \text{among } E \subset \Omega,$$

since one needs $\kappa \geq h(\Omega)$;

(v) it acts as threshold for the existence of solutions of

$$\operatorname{div} \frac{\nabla u(x)}{\sqrt{1 + |\nabla u(x)|^2}} = H, \qquad x \in \Omega,$$

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Robustness of the relations

Most of these relations hold in very general spaces. Given a metric measure space $(X, \mathcal{B}, \mathfrak{m})$, and $P(\cdot)$ the perimeter induced by the metric, one can define for $\Omega \subset X$

$$h(\Omega) := \inf \left\{ \frac{P(E)}{\mathfrak{m}(E)}, \, E \subset \Omega, \, \mathfrak{m}(\Omega) > 0 \right\}.$$

If $(X, \mathcal{B}, \mathfrak{m})$ is "good" one retains the spectral relations we mentioned.

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Today's plan

Today we shall zero-in on the easiest possible framework, that of the Euclidean N-dimensional space, for an open, and bounded set Ω.

A brief overview of our tour:

1) we start in dimension $N = 2$;
2) we pass to estimates for cylindrical sets;
3) we conclude with quantitive estimates.

$N = 2$: convex sets and strips

Given $\Omega \subset \mathbb{R}^2$ its "inner parallel set" at distance t is

$$\Omega^t := \{x \in \Omega \,:\, \mathrm{dist}(x, \partial\Omega) \geq t\}.$$

Let Ω be convex [Kawohl & Lachand-Robert (2006)] or a strip [Leonardi & Pratelli (2016)]. Then, Ω has a unique Cheeger set E given by

$$E = \bigcup_{x \in \Omega^r} B_r(x),$$

where $r = 1/h(\Omega)$. Moreover, the inner Cheeger formula holds: r is the unique positive solution of

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$N = 2$: sets w/no necks

Definition

A set Ω has no "necks" of radius r if the following holds. Given two balls of radius r, $B_r(x_0)$ e $B_r(x_1)$ contained in Ω, there exists a continuous curve $\gamma : [0,1] \to \Omega$, such that

$$\gamma(0) = x_0, \qquad \gamma(1) = x_1, \qquad \text{and} \qquad B_r(\gamma(t)) \subset \Omega\,, \qquad \forall t \in [0,1].$$

Theorem (Leonardi, Neumayer, S. (2017) & Leonardi, S. (2020))

Let Ω be a Jordan domain such that $|\partial\Omega| = 0$, and let r be the unique positive solution of $\pi t^2 = |\Omega^t|$. If Ω has no necks of radius r, then

$$h(\Omega) = \frac{1}{r}, \qquad \textit{and} \qquad E = \bigcup_{x \in \Omega^r} B_r(x),$$

is the maximal Cheeger set of Ω.

An unfortunate example

The theorem in principle is great. It gives a formula to compute the constant and a recipe to find a Cheeger set. At times though..

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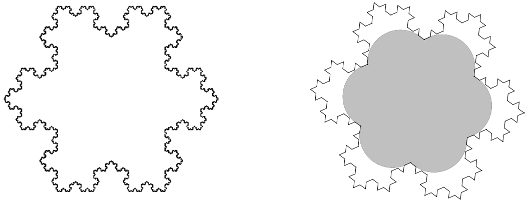

A Koch snowflake has no necks of any size, thus the inner Cheeger formula holds. Yet, this is not algebraically solvable.

Approximating $h(\Omega)$

We have an error estimate of $h(\Omega)$ w.r.t. interior approximations.

Proposition (Leonardi, Neumayer, S. (2017))

Let ω, Ω be open sets in $\mathbb{R}^2$ such that $\omega \subset \Omega$. Let $r = h^{-1}(\omega)$ and $R = h^{-1}(\Omega)$ and assume the inner Cheeger formula holds for both ω, Ω, i.e., $|\omega^r| = \pi r^2$ and $|\Omega^R| = \pi R^2$. Then,

$$0 \le R - r \le \frac{|\Omega^r \setminus \omega^r|}{2\pi r}.$$

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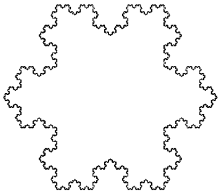

Computing r_n for the approximating sets is easy. The error is

$$0 \le r_\infty - r_n \le \frac{2^{n-3}}{25 \cdot 3^{3n-5}}\,.$$

Estimates for cylindrical sets

Fix the cylinder $\Omega \subset \mathbb{R}^{N+1}$ w/cross section $\omega \subset \mathbb{R}^N$ & height $L > 0$

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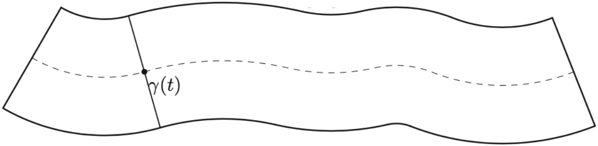
If $N = 1$, cylinders are rectangles, which we got covered. Similarly to rectangles, one can consider "strips" [Krejčiřík & Pratelli (2011)],



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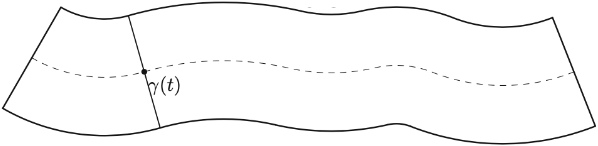


and have (here ω is a segment $[0, \ell]$)

$$h(\omega) + \frac{1}{400}\frac{1}{L} \leq h(\Omega) \leq h(\omega) + \frac{2}{L}.$$

A second order improvement came with [Leonardi & Pratelli (2016)].

Estimates for cylindrical sets

Similar estimates hold general N-dimensional cross section $\omega \subset \mathbb{R}^N$.

Theorem (Pratelli, S. (forthcoming))

Let $\Omega = \omega \times [0, L]$ w/cross section $\omega \subset \mathbb{R}^N$ & height $L > 0$. There exists a constant $c = c(\omega) > 0$, such that

$$h(\omega) + \frac{c}{L} \le h(\Omega) \le h(\omega) + \frac{2}{L}.$$

The constant c only depends on the volume of ω's "smallest" Cheeger set.

An application

Let $p > 1$, and let $F_{p,1}$ be the shape functional

$$F_{p,1}[E] := \frac{\lambda_p(E)}{h^p(E)} .$$

defined over $\mathcal{K}^N$, the class of convex subsets of $\mathbb{R}^N$.

Theorem (Pratelli, S. (forthcoming))

There exist minima of $F_{p,1}$ in $\mathcal{K}^N$, in any dimension N.

This positively solves a conjecture by [Parini (2017)] (for $p = 2$) and by [Briani, Buttazzo, Prinari (2021)], exploiting the estimates for cylindrical sets and tools proved in [Ftouhi (2021)].

Quantitative estimates

Let us recall the isoperimetric quantitative inequalities. Given any set Ω, we let B_Ω be the ball with its same volume. We have

(i) $\frac{P(\Omega)-P(B_\Omega)}{P(B_\Omega)} \geq 0$,

(ii) $\frac{P(\Omega)-P(B_\Omega)}{P(B_\Omega)} \geq c\alpha^2(\Omega)$,

(iii) $\frac{P(\Omega)-P(B_\Omega)}{P(B_\Omega)} \geq c\zeta(\Omega)$,

(iv) $\frac{P(\Omega)-P(B_\Omega)}{P(B_\Omega)} \geq c\beta^2(\Omega)$,

where $\alpha(\Omega)$, $\zeta(\Omega)$ and $\beta(\Omega)$ are asymmetry indexes, measuring the distance in some suitable sense of Ω from an optimal set.

Can one obtain the analog with $h(\cdot)$ in place of $P(\cdot)$?

Quantitative estimates

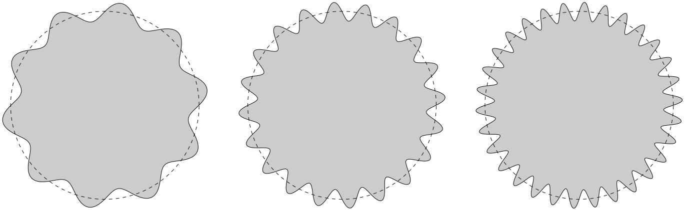
The analogue of (i) is an immediate consequence of (i) itself, while the analogue of (ii) has been proved in [Figalli, Maggi, Pratelli (2009)].

Theorem (Julin, S. (2021))

Given $\Omega \subset \mathbb{R}^N$ open and bounded. There exists $c = c(N)$ such that

$$\frac{h(\Omega) - h(B_\Omega)}{h(B_\Omega)} \geq c\zeta(\Omega)\,.$$

It does not exist $c = c(N)$ s.t. the inequality with β^2 in place of ζ holds.



Quantitative Gaussian estimates

We have also proved similar estimates in the Gaussian space, where the volume and the perimeter are weighted by the Gaussian measure γ.

Theorem (Julin, S. (2021))

Given $\Omega \subset \mathbb{R}^N$ open and bounded. There exist $c_1 = c_1(\gamma(\Omega))$ such that

$$h_\gamma(\Omega) - h_\gamma(H_\Omega) \geq c_1 \alpha_\gamma^2(\Omega).$$

and $c_2 = c_2(\gamma(\Omega))$ such that

$$h_\gamma(\Omega) - h_\gamma(H_\Omega) \geq c_2 \frac{\beta_\gamma(\Omega)}{1 + \sqrt{|\log(\beta_\gamma(\Omega))|}}.$$

References

- Julin, V. and Saracco, G., Quantitative lower bounds to the Euclidean and the Gaussian Cheeger constants, *Ann. Fenn. Math.* 46:2, 2021
- Leonardi, G. P., Neumayer, R. and Saracco, G., The Cheeger constant of a Jordan domain without necks, *Calc. Var. Partial Differential Equations* 56(6):164, 2017
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