Charged Liquid Drop

Domenico Angelo La Manna

domenicolamanna@hotmail.it

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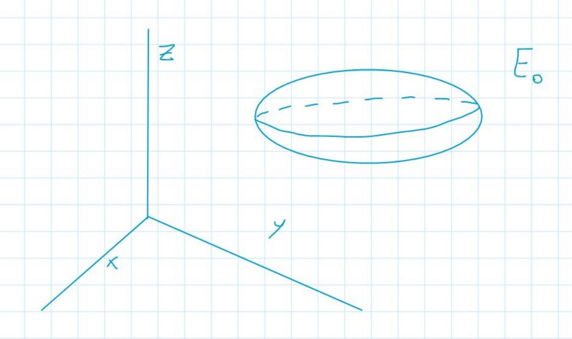
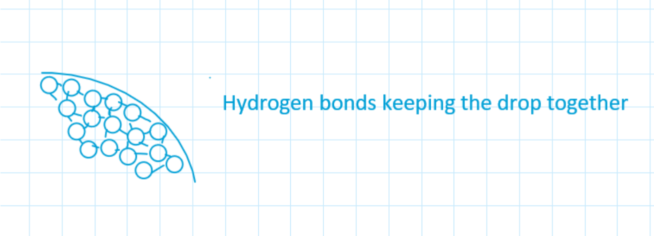
Summary I

Introduction

Assume we are looking at a charged liquid drop in the vacuum and we want to understand its dynamics. At the initial time we think to the droplet as a set $E_0 \subset \mathbb{R}^3$

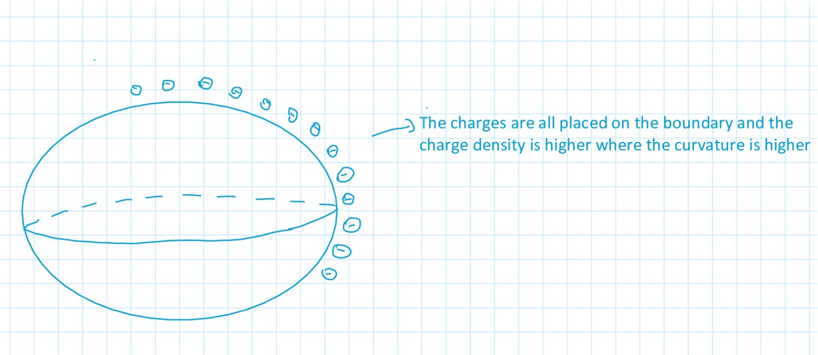


Surface tension



The surface tension act on the surface of the droplet and its role is to keep the droplet together.

The Electric field



Since the density charge is higher where the curvature is higher, we have that there will be more charged particles repelling each other in correspondence of very curved piece of the surface.

State of the art

- The stationary problem has been intensively studied in the past years (Figalli, Fusco, Julin, Knupfer, Maggi, Muratov, Novaga, Otto, Ruffini and so on).
- When $Q = 0$, many results are already available about the evolution of the drop: Coutand and Shkoller have proven the short time existence for the system while Shatan and Zeng proved some a priori estimates.
- Independently from Shatan and Zeng, also Schweizer proved a priori regularity assuming that the drop can be seen as a graph.
- About free boundary Euler equation it is worth to mention an important paper of Lindblad. In his work he studied the case where the boundary equation is just $p = 0$ and he proves the well posedness in case $-\partial_\nu p \geq c_0 > 0$.

The equation

We think at the drop at initial time as a set $E_0 \subset \mathbb{R}^3$ and assume we know the initial velocity of each particle, let us call it $v_0 : E_0 \to \mathbb{R}^3$. The unknown of the problem are the velocity field, the evolving shape and the pressure.

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$$\begin{cases} \mathcal{D}_t v + \nabla p = 0 \qquad \text{in } E_t \\ \operatorname{div} v = 0 \qquad \text{in } E_t \\ v_n = V_t \qquad \text{on } \partial E_t \\ p = H_{E_t} - \frac{Q}{2C_t^2}|\nabla U_{E_t}|^2 \qquad \text{on } \partial E_t, \end{cases} \tag{2.1}$$

Notation

In the previous system we used the notation

- with $\mathcal{D}_t$ we denote the differential operator $v \to \mathcal{D}_t v = \partial_t v + \langle v, \nabla \rangle v$ and we will refer to it as material derivative;
- $v(x,t) : E_t \to \mathbb{R}^3$ is the velocity field and it gives the velocity of a particle $x \in E_t$ (Eulerian point of view);
- H_{E_t} is the mean curvature of E_t;
- U_{E_t} stands for the capacitary potential of E_t and $C_t = \text{cap}(E_t)$;
- $v_n = \langle v, \nu_{E_t} \rangle$ is the normal component of the velocity field at a point $x \in \partial E_t$ and V_t is the normal velocity of the fluid at $x \in \partial E_t$.

Conservation law

Observe that the equation div $v = 0$ implies that there is mass conservation

$$\frac{d}{dt}|E_t| = \int_{E_t} \operatorname{div} v \, dx = 0.$$

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Secondly, we note that since the system is conservative, we also have the conservation of the total energy

$$\frac{d}{dt}\left(\underbrace{\frac{1}{2}\int_{E_t} |v|^2 \, dx}_{\text{Kinetic energy}} + \underbrace{P(E_t) + \frac{Q}{2\mathrm{cap}(E_t)}}_{\text{Potential energy}} \right) = 0$$

Framework and a priori assumptions

We parametrize the moving boundary $\Sigma_t = \partial E_t$ with a fixed reference surface Σ which is smooth and compact. As usual we use the height function parametrization which means that

$$\Sigma_t = \{h(x,t)\nu_\Sigma + x : x \in \Sigma\}.$$

We assume that Σ satisfies the interior and exterior ball condition with radius $\eta > 0$.
Our a priori assumption read as

$$\Lambda_T := \sup_{t\in(0,T]} \Big(\|h(\cdot,t)\|_{C^{1,\alpha}(\Sigma)} + \|\nabla v(\cdot,t)\|_{L^\infty(E_t)} + \|v_n(\cdot,t)\|_{H^2(\Sigma_t)} + \|p(\cdot,t)\|_{L^1(\Sigma_t)}\Big) < \infty. \tag{3.1}$$

The main theorem

Main Theorem

Assume that E_0 is a smooth initial set and let $v_0 \in C^\infty(E_0)$ be the initial velocity field. Assume that the system (2.1) *has a classical solution in time-interval* $(0, T)$ *and the parametrization satisfies* (3.1)*. Then there is $\delta > 0$, which depends on T, Λ_T, $\|v_0\|_{H^3(E_0)}$ and on $\|H_{\Sigma_0}\|_{H^2(\Sigma_0)}$, such that the solution exists in a time interval* $(0, T + \delta]$.

Main Theorem, part two

Main Theorem

The solution is smooth and for every $l \geq 1$ it holds

$$\sup_{t\in(0,T+\delta)} \sum_{k=0}^{l} \|\mathcal{D}_t^{l+1-k} v\|^2_{H^{\frac{3}{2}k}(E_t)} + \|v(\cdot,t)\|^2_{H^{\lfloor\frac{3}{2}(l+1)\rfloor}(E_t)}. \leq C_l.$$

The constant C_l depends on l, T, σ_T, Λ_T and on the initial datum. In particular, the system (2.1) has a smooth solution as long as the a priori estimates (3.1) hold.
Moreover, there is $T_0 > 0$, which depends on $\|v_0\|_{H^3(E_0)}$ and on $\|H_{\Sigma_0}\|_{H^2(\Sigma_0)}$, such that the a priori estimates (3.1) hold for $T = T_0$.

Comments on the main theorem

- Even though a priori estimates when the drop is uncharged are already present in literature, our result improves the existing results on the subject.
- From the point of view of the shape of the drop, we deduce that if the parametrization of the flow remains $C^{1,\alpha}$-regular then the flow does not create singularities
- We do not know if the assumption of v being Lipschitz and $v_n \in H^2(\Sigma_t)$ are sharp.
- Our main contribution to the problem is to find lower order sufficient conditions which guarantees that the flow is well-defined and to provide regularity estimates.
- Out main result implies that the system (2.1) has a solution in $(0, T_0)$ which is as regular as the initial data.

Main challanges

- It is well known that interpolation inequalities on a two dimensional manifold Σ hold true when its curvature is in $L^{2+\varepsilon}$. Since in (3.1) we only assume $p \in L^1$ (which more or less implies $H_{\Sigma_t} \in L^1$) this means that we do not have enough regularity to use such an important tool straight forward. Nevertheless, we are able to prove that our Assumptions implies that the curvature is actually in L^4;
- Once we have proven that the curvature is in L^4 we have to prove sharp elliptic estimates (for functions and vector fields) for non non homogeneous problems with low boundary regularity. For instance, we proved that for a vector field V it holds

$$\|V\|_{H^1(E)} \le C(\|V_n\|_{H^{\frac{1}{2}}(\Sigma)} + \|V\|_{L^2(E)} + \|\operatorname{div} V\|_{L^2(E)} + \|\operatorname{curl} V\|_{L^2(E)})$$

Main challanges

- Since the boundary regularity is part of the energy in some sense, it is important that in all the estimates the dependence on the curvature of the constants is explicit ;
- When proving higher order regularity, the choice of the energy to differentiate it is not obvious. One of the reason is that the velocity field v and the capacitary potential U_t live in different regions, hence it is not very safe to take a material derivative of U_t;
- Since our goal is to find linear energy estimate, so we always need to be very careful with computation. In fact, there will be many terms piling up for which a careful analysis made with the sharp Kato-Ponce inequality is necessary;
- When differentiating with respect to time the energy functional, we have to deal with many object whose expression is very complicated .

Quantities of interest

To prove our result, for $l \in \mathbb{N}$ we introduce the quantities

$$\mathcal{E}_l(t) = \int_{E_t} |\mathcal{D}_t^{l+1} v|^2 \, dx + \int_{\Sigma_t} |\bar{\nabla}(\mathcal{D}_t^l v \cdot \nu)|^2 \, d\mathcal{H}^2 - \int_{E_t^c} |\nabla(\partial_t^{l+1} U)|^2 \, dx + \int_{E_t} |\nabla^{\lfloor \frac{1}{2}(3l+1) \rfloor} \omega|^2 \, dx ,$$

which can be seen as higher order energies, and

$$E_l(t) := \sum_{k=1}^{l} \|\mathcal{D}_t^{l+1-k} v\|^2_{H^{\frac{3}{2}k}(E_t)} + \|v\|^2_{H^{\lfloor \frac{3}{2}(l+1) \rfloor}(E_t)} + \|\mathcal{D}_t^l v \cdot \nu\|^2_{H^1(\Sigma_t)} + 1.$$

Case $l = 1$

At the first stage (i.e. the case $l = 1$) the proof is much harder since we do not have much regularity of the boundary and we need to prove it. Hence as a starter, we note that pressure bounds imply curvature bounds, hence regularity of the boundary.

Lemma 1

Assume that (3.1) *holds. Then*

$$\|B\|_{L^4(\Sigma_t)} \leq \varepsilon \|p\|_{H^1(\Sigma_t)} + C_\varepsilon$$

and

$$\|B\|_{H^1(\Sigma_t)} \leq C(1 + \|p\|_{H^1(\Sigma_t)}).$$

Case $l = 1$

Secondly we prove the following key estimates

Proposition 2

Assume that the apriori estimate (3.1) *holds for* $T > 0$. *Then*

$$\sup_{t\in(0,T]} \|\nabla p\|^2_{L^2(E_t)} \le e^{CT}(1 + \|\nabla p\|^2_{L^2(E_0)}).$$

Proposition 3

Assume that the apriori estimates (3.1) *holds for* $T > 0$ *and denote* $\tilde{\Lambda}_T := \sup_{t\in(0,T]} \|\nabla p\|^2_{L^2(E_t)}$. *Then*

$$\int_0^T \|p\|^2_{H^2(E_t)}\, dt \le C(1 + T).$$

Case $l = 1$

The next step is to prove the following

Proposition 4

Assume that the apriori estimate (3.1) *holds for* $T > 0$*. Then*

$$\frac{d}{dt}\mathcal{E}_1(t) \leq C(1 + \|p\|^2_{H^2(E_t)})(1 + E_1(t)),$$

The proof of this proposition is harder than it looks: it relies on very careful estimates of the integrals appearing when differentiating and a series of regularity theorems (i.e. harmonic estimates, trace theorem, interpolation inequalities and so on) with low boundary regularity that we had to prove by ourselves.

Case $l = 1$

The next proposition is essentially stating that in terms of regularity, one material derivatives corresponds to $3/2$ space derivatives.

Proposition 5

Assume that the apriori estimate (3.1) *holds for* $T > 0$. *Then there exists constants* C_0 *such that*

$$E_1(t) \leq C(C_0 + \mathcal{E}_1(t)).$$

where the constants depends on Λ_T *and* $\tilde{\Lambda}_T$ *defined in Proposition 3.*

Higher order estimates

Proposition 6

Let $l \geq 2$ and assume that (3.1) and $E_{l-1}(t) \leq C$ hold for all $t \in [0, T]$. Then it holds

$$\frac{d}{dt}\mathcal{E}_l(t) \leq CE_l(t).$$

Proposition 7

Let $l \geq 2$ and assume that (3.1) and $E_{l-1}(t) \leq C$ hold for all $t \in [0, T]$. Then there are constants C_1 and C_2 such that

$$E_l(t) \leq C_1\mathcal{E}_l(t) + C_2.$$

Energy estimates imply regularity

Assuming that (3.1) hold, by Proposition 4 and Proposition 5 we have

$$\frac{d}{dt}\mathcal{E}_1(t) \leq C(1 + \|p\|^2_{H^2(E_t)})E_1(t)$$
$$\leq C(1 + \|p\|^2_{H^2(E_t)})(C_0 + \mathcal{E}_1(t)).$$

This, together with Proposition 3 in turn implies

$$C_0 + \mathcal{E}_1(t) \leq C(C_0 + \mathcal{E}_1(0))e^{CT}$$

which also gives

$$E_1(t) \leq C(C_0 + \mathcal{E}_1(t)) \leq C(C_0 + \mathcal{E}_1(0))e^{CT} \leq CE_1(0)e^{CT}.$$

Hence we prove that $E_1(t)$ stays finite as long as (3.1) holds.

Energy estimates imply regularity

We may then use Proposition 6 and Proposition 7 in an inductive way and deduce that if $E_{l-1}(t) \leq C$ for $t \leq T$ then it holds

$$\frac{d}{dt}\mathcal{E}_l(t) \leq CE_l(t) \leq C(C_0 + \mathcal{E}_l(t)).,$$

By integrating we deduce

$$C_0 + \mathcal{E}_l(t) \leq C(C_0 + \mathcal{E}_l(0))e^{CT}$$

and using Proposition 7 again implies

$$E_l(t) \leq C(C_0 + \mathcal{E}_l(0))e^{CT} \leq CE_l(0)e^{CT},$$

where the constants depend now also on l and on $E_{l-1}(0)$. Therefore an induction argument implies that if $E_l(T) < C$

Some computation

In order to make a bit more clear what we have said, we scketch the proof of Proposition 4. From the equations $\mathcal{D}_t v = -\nabla p$ and $p = H_{\Sigma_t} - Q|\nabla U_t|^2/(2C_t)$ we obtain

$$\mathcal{D}_t^3 v = -\nabla \mathcal{D}_t^2 p + [\mathcal{D}_t^2, \nabla] p,$$

$$\mathcal{D}_t^2 p = -\Delta_\Sigma \langle \mathcal{D}_t v, \nu \rangle + \partial_\nu U \partial_\nu \partial_t^2 U + \mathrm{Err}.$$

and since $U_t = 1$ on Σ_t we also have

$$\partial_t U_t = -\langle \nabla U_t, v \rangle = \partial_\nu U_t v_n$$

With these equations in mind we can compute the derivative of the energy

Some computation

$$\frac{d}{dt}\frac{1}{2}\int_{E_t}|\mathcal{D}_t^2 v|\,dx = \int_{E_t}\langle \mathcal{D}_t^3 v, \mathcal{D}_t^2 v\rangle\,dx =$$

$$= -\int_{E_t}\langle \nabla \mathcal{D}_t^2 p, \mathcal{D}_t^2 v\rangle\,dx + \int_{E_t}\langle [\mathcal{D}_t^2, \nabla]p, \mathcal{D}_t^2 v\rangle\,dx$$

$$= -\int_{\Sigma_t}\mathcal{D}_t^2 p\,\langle \mathcal{D}_t^2 v, \nu\rangle\,d\mathcal{H}^2 + \int_{E_t}\mathcal{D}_t^2 p\,\mathrm{div}(\mathcal{D}_t^2 v)\,dx + \mathrm{Err}_1.$$

Some computation

$$
\begin{aligned}
&\frac{d}{dt}\frac{1}{2}\int_{\Sigma_t} |\nabla_\tau(\mathcal{D}_t v\cdot\nu)|^2\, d\mathcal{H}^2 \\
&= \int_{\Sigma_t} \langle \mathcal{D}_t\nabla_\tau(\mathcal{D}_t v\cdot\nu), \nabla_\tau(\mathcal{D}_t v\cdot\nu)\rangle\, d\mathcal{H}^2 \\
&\quad + \frac{1}{2}\int_{\Sigma_t} |\nabla_\tau(\mathcal{D}_t v\cdot\nu)|^2 (\mathrm{div}_\tau\, v)\, d\mathcal{H}^2 \\
&= -\int_{\Sigma_t} (\Delta_{\Sigma_t}(\mathcal{D}_t v\cdot\nu))(\mathcal{D}_t^2 v\cdot\nu)\, d\mathcal{H}^2 + \mathrm{Err}_2
\end{aligned}
$$

Some computation

$$
\begin{aligned}
&-\frac{d}{dt}\frac{1}{2}\int_{E_t^c}|\nabla\partial_t^2 U_t|^2\,dx\\
&=-\int_{E_t^c}\langle\nabla\partial_t^3 U_t,\nabla\partial_t^2 U\rangle\,dx+\frac{1}{2}\int_{\Sigma_t}|\nabla\partial_t^2 U_t|^2\,d\mathcal{H}^2\\
&=\int_{\Sigma_t}(\partial_\nu U_t\,\partial_\nu\partial_t^2 U_t)(\mathcal{D}_t^2 v\cdot\nu)\,d\mathcal{H}^2+\mathrm{Err}_3,
\end{aligned}
$$

and

$$
\begin{aligned}
\frac{d}{dt}\frac{1}{2}\int_{E_t}|\nabla^2\omega|^2\,dx &= \int_{E_t}\nabla v\star\nabla^2\omega\star\nabla^2\omega\,dx+\|\nabla^2\omega\|_{L^2}^2\\
&\le\|\nabla^2\omega\|_{L^2(E_t)}^2+\mathrm{Err}_4
\end{aligned}
$$

Some computation

After estimating the error terms we arrive at

$$\frac{d}{dt}\frac{1}{2}\mathcal{E}_1(t) \leq C(1+\|p\|_{H^2(E_t)^2})E_1(t) + \int_{E_t} \mathcal{D}_t^2 p \,\mathrm{div}(\mathcal{D}_t^2 v)\,dx.$$

For the last term we estimate as

$$\int_{E_t} \mathcal{D}_t^2 p \,\mathrm{div}(\mathcal{D}_t^2 v)\,dx \leq \|\mathcal{D}_t^2 p\|_{H^{-\frac{1}{2}}(E_t)}\|\mathrm{div}(\mathcal{D}_t^2 v)\|_{H^{\frac{1}{2}}(E_t)} \leq C(1+\|p\|_{H^2(E_t)^2})E_1(t).$$

which proves Proposition 4.

Thank you for you attention