ISOPERIMETRIC INEQUALITIES AND POTENTIAL THEORY

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Isoperimetric Problems

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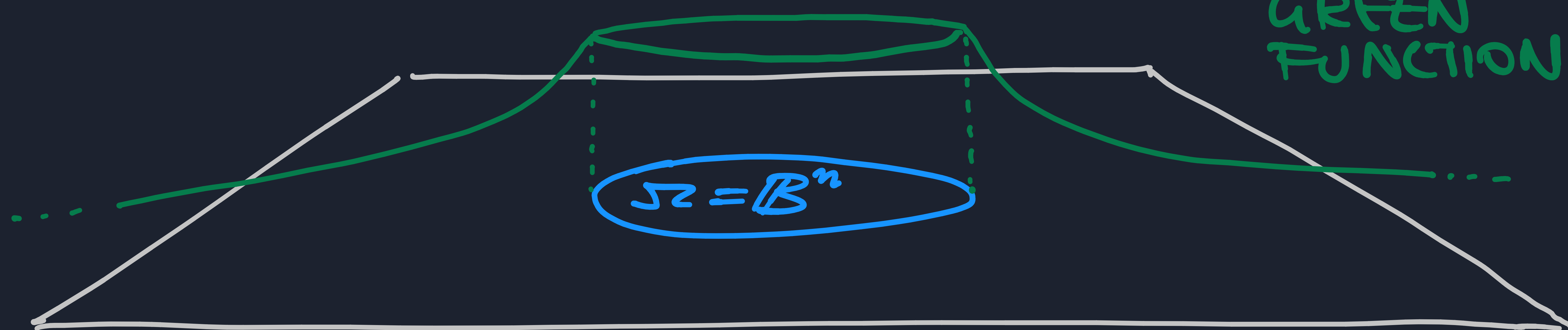
1) LINEAR POTENTIAL THEORY in $\mathbb{R}^n$

$$\begin{cases} \Delta u = 0 \,, & \mathbb{R}^n \setminus \overline{\Omega} \\ u = 1 \,, & \partial\Omega \\ u \to 0 \,, & \text{as } |x| \to +\infty \end{cases}$$

smooth bounded domain

Model solution:



$u = |x|^{2-n}$

GREEN FUNCTION

$\boxed{\text{DEF}}$

$$F_\beta : [1, +\infty) \longrightarrow \mathbb{R} \quad , \beta \in \mathbb{R}$$

$$F_\beta(\tau) := \tau^{\beta\left(\frac{n-1}{n-2}\right)} \cdot \int_{\{u = 1/\tau\}} |Du|^{\beta+1} \, d\sigma$$

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$$= \int_{\{u = 1/\tau\}} \left(\frac{|Du|}{u^{\frac{n-1}{n-2}}} \right)^{\beta+1} \cdot u^{\frac{n-1}{n-2}} \, d\sigma$$

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CONSTANT ON
ROT. SYMM. SOLUTIONS

$$= \int_{\{u=1/\tau\}} \left(\frac{|Du|}{u^{\frac{n-1}{n-2}}} \right)^{\beta+1} \cdot u^{\frac{n-1}{n-2}} \, d\sigma$$

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$\Longrightarrow$ F_β IS CONSTANT ON ROT. SYMM. SOLUTIONS !

To compute the value of F_β at $+\infty$ recall:

- $$\mathcal{C}_2(\Omega) = \inf \left\{ \frac{\int_{\mathbb{R}^n} |Dw|^2 \, d\mu}{(n-2)|\mathbb{S}^{n-1}|} \;\middle|\; \begin{array}{l} w \in C_c^\infty(\mathbb{R}^n) \\ w \equiv 1 \text{ on } \Omega \end{array} \right\}$$

NORMALIZED 2-CAPACITY

$$= \frac{1}{(n-2)|\mathbb{S}^{n-1}|} \int_{\partial\Omega} |Du| \, d\sigma$$

- Asymptotic expansion:

$$u(x) = \mathcal{C}_2(\Omega) \cdot |x|^{2-n} + \mathcal{O}_2(|x|^{1-n})$$

as $|x| \to +\infty$

It follows that :

$$F_\beta(\tau) \longrightarrow \left[C_2(\Omega)\right]^{\frac{n-2-\beta}{n-2}} \cdot (n-2)^{\beta+1} |S^{n-1}|$$

as $\tau \to +\infty$.

$\boxed{\text{THEOREM}}$ (Agostiniani, ——— 2016)

$$\forall\, \beta \geq \left(\frac{n-2}{n-1}\right) \;:\; F_\beta \in \mathcal{C}^1\big([1,+\infty)\big)$$

&

$$\boxed{F_\beta'(\tau) \leq 0} \,.$$

Moreover, the monotonicity is strict unless Ω is a ball.

Smooth computations: (with $|Du| \neq 0$)

$$F_\beta'(\tau) = -\beta\, \tau^{\beta\left(\frac{n-2}{n-1}\right)-2} \int_{\{u=1/\tau\}} |Du|^\beta \left[H - \left(\frac{n-1}{n-2}\right) \left|\frac{Du}{u}\right| \right] d\sigma$$

By the asymptotic expansions one has

$$\lim_{\tau \to +\infty} F_\beta'(\tau) = 0$$

$$
F_\beta''(\tau) = \beta \cdot \tau^{\beta\left(\frac{n-1}{n-2}\right)-4} \cdot \int_{\{u=1/\tau\}} |Du|^{\beta-1} \times \left\{ \beta \left| \frac{D^T |Du|}{|Du|} \right|^2 + |\mathring{h}|^2 + \mathrm{Ric}\left(\frac{Du}{|Du|}, \frac{Du}{|Du|} \right) + \left(\beta - \frac{n-2}{n-1} \right) \left[H - \frac{n-1}{n-2} \left| \frac{Du}{u} \right| \right]^2 \right\} d\sigma
$$

Application:

$$\lim_{\tau \to +\infty} F_\beta(\tau) = \left[c_2(\Omega) \right]^{\frac{n-2-\beta}{n-2}} \cdot (n-2)^{\beta+1} \cdot |\mathbb{S}^{n-1}|$$

$\wedge$ ← THEOREM

$$F_\beta(1) = \int_{\partial\Omega} |Du|^{\beta+1} \, d\sigma \leq (n-2)^{\beta+1} \int_{\partial\Omega} \left| \frac{H}{n-1} \right|^{\beta+1} d\sigma$$

$$0 \leq -F_\beta'(1) = \beta \int_{\partial\Omega} |Du|^\beta \left[H - \left(\frac{n-1}{n-2} \right) |Du| \right] d\sigma$$

+ HÖLDER INEQ.

with $\beta = n-2$ one gets

$\boxed{\text{COROLLARY}}$ (Willmore-type Ineq.)

$$|\mathbb{S}^{n-1}| \leq \int_{\partial\Omega} \left|\frac{H}{n-1}\right|^{n-1} d\sigma$$

with $\beta = n-2$ one gets

$\boxed{\text{COROLLARY}}$ (Willmore-type Ineq.)

$$|\mathbb{S}^{n-1}| \leq \int_{\partial\Omega} \left|\frac{H}{n-1}\right|^{n-1} d\sigma$$

and $=$ $\iff$ $\partial\Omega$ is a sphere of radius

$$[C_2(\Omega)]^{1/n-2}$$

2) LINEAR POTENTIAL THEORY & RICCI BOUNDS

(M, g) complete non cpt. Riem. mfld

- $\mathrm{Ric}_g \geq 0$
- Euclidean volume growth

$0 < \mathrm{AVR}(g) \leq 1$

(EX. Asympt. conical ; A.L.E. with $\mathrm{Ric} \geq 0$; GRAV. ISTANTONS)

$\boxed{\text{THEOREM}}$ (Agostiniani, Fogagnolo, —— 2018)

$$\mathrm{AVR}(g)\, |\mathbb{S}^{n-1}| \leq \int_{\partial\Omega} \left| \frac{H}{n-1} \right|^{n-1} d\sigma$$

Rigidity:

$\textcircled{=} \iff (M \setminus \overline{\Omega}, g) \underset{\text{isom.}}{\cong}$ 

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$\boxed{\text{COROLLARY}}$ (3D - Isop. Ineq.)

$$\inf_{\Omega \subseteq M^3} \frac{|\partial\Omega|^3}{36\pi|\Omega|^2} = \inf_{\Omega \subseteq M^3} \frac{\int_{\partial\Omega} H^2 \, d\sigma}{16\pi} = \mathrm{AVR}(g)$$

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Hence:

$$\boxed{\underset{\text{ISOPERIMETRIC}}{36\pi\,\mathrm{AVR}(g)} \leq \frac{|\partial\Omega|^3}{|\Omega|^2}}$$

Huisken's computation:

- $\{\partial\Omega_t\}_{t\in[0,T)}$ Mean Curvature Flow starting at $\partial\Omega_0 = \partial\Omega$

$$\boxed{D(t) = |\partial\Omega_t|^{3/2} - K\,|\Omega_t|}$$

- $\dfrac{d}{dt}|\partial\Omega_t| = -\displaystyle\int_{\partial\Omega_t} H^2\,d\sigma \quad ; \quad \dfrac{d}{dt}|\Omega_t| = -\displaystyle\int_{\partial\Omega_t} H\,d\sigma$

- $\dfrac{d}{dt}D = -\dfrac{3}{2}|\partial\Omega_t|^{1/2}\displaystyle\int_{\partial\Omega_t} H^2\,d\sigma + K\int_{\partial\Omega_t} H\,d\sigma \leq$

$$\leq -\frac{3}{2}|\partial\Omega_t|^{1/2}\int_{\partial\Omega_t} H^2\,d\sigma + K\left(\int_{\partial\Omega_t} H^2\,d\sigma\right)^{1/2}\cdot|\partial\Omega_t|^{1/2} =$$

$$\ldots = |\partial\Omega_t|^{1/2} \cdot \left(\int_{\partial\Omega_t} H^2\, d\sigma\right)^{1/2} \cdot \left[-\frac{3}{2}\left(\int_{\partial\Omega_t} H^2\, d\sigma\right)^{1/2} + K\right] \leqslant$$

$\boxed{\text{WILLMORE}}$

$$\leqslant |\partial\Omega_t|^{1/2} \cdot \left(\int_{\partial\Omega_t} H^2\, d\sigma\right)^{1/2} \cdot \left[-\frac{3}{2}\sqrt{16\pi\, AVR(g)} + K\right] =$$

$$= |\partial\Omega_t|^{1/2} \cdot \left(\int_{\partial\Omega_t} H^2\, d\sigma\right)^{1/2} \cdot \left[K - \sqrt{36\pi\, AVR(g)}\right] = 0$$

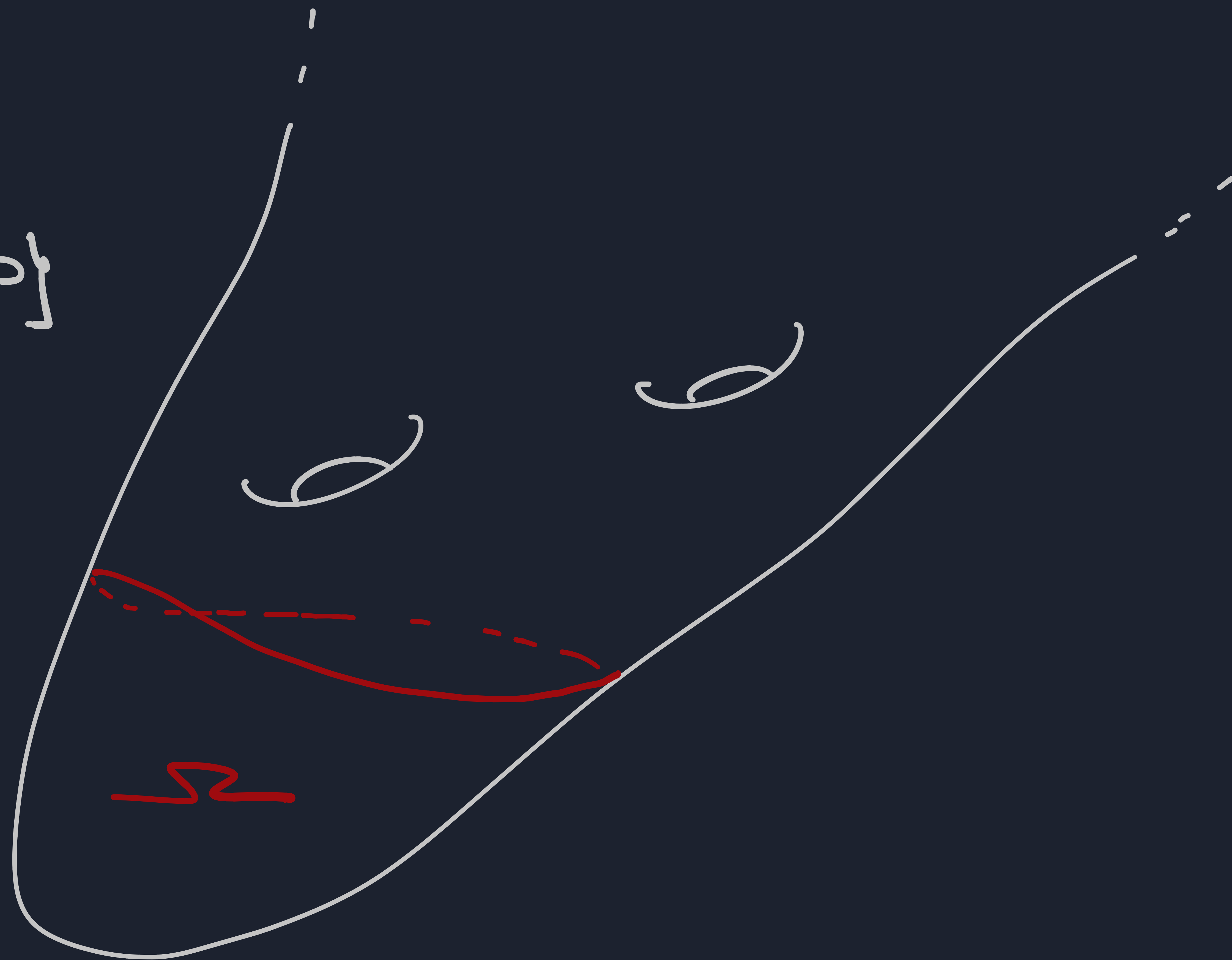
$$K = \sqrt{36\pi\, AVR(g)}$$

Hence:

$$0 = D(T^-) \leq D(0) = |\partial\Omega|^{3/2} - \sqrt{36\pi\, AVR(g)}\, |\Omega|$$

Rmk

Letting $\Omega \to \{p\}$



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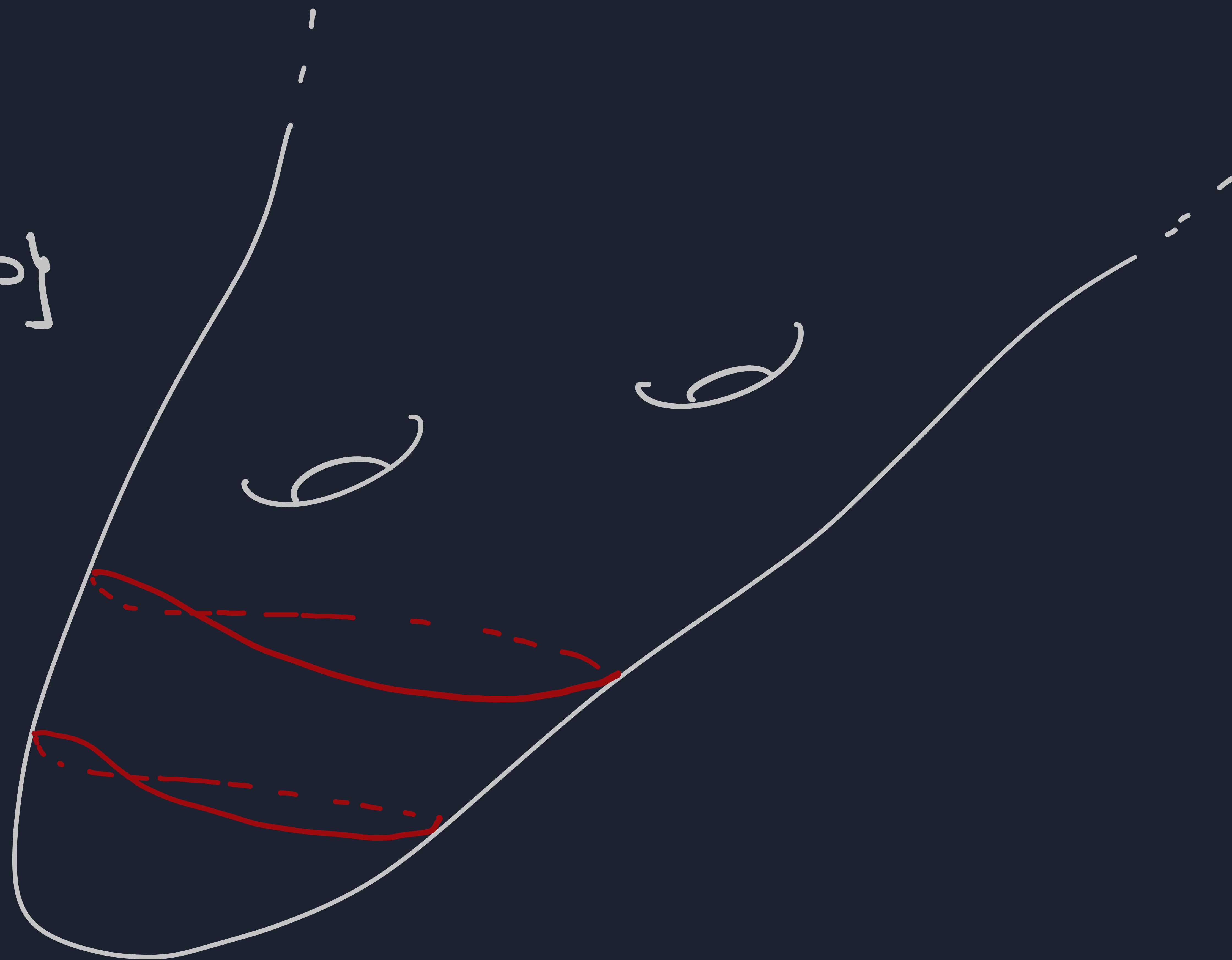

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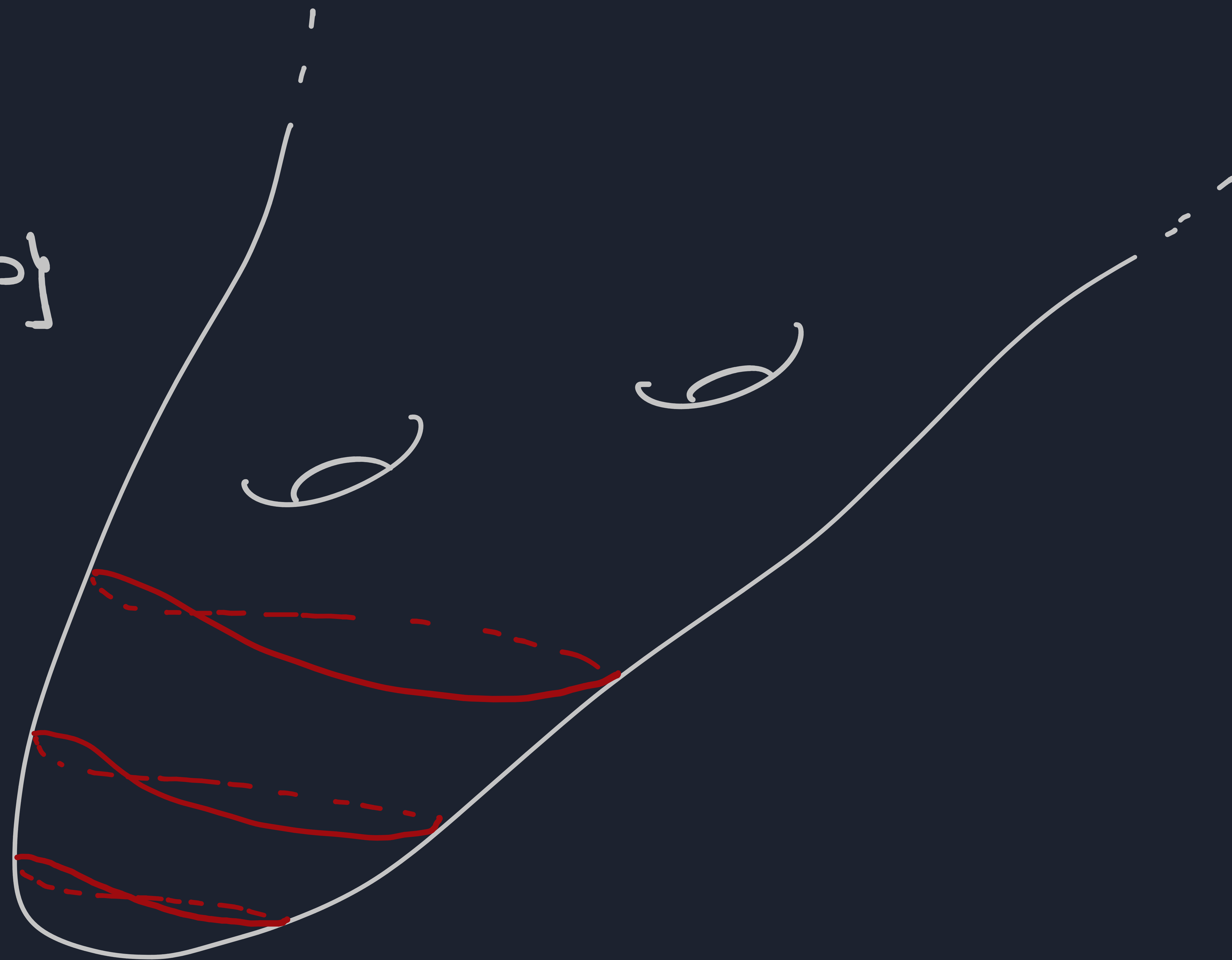


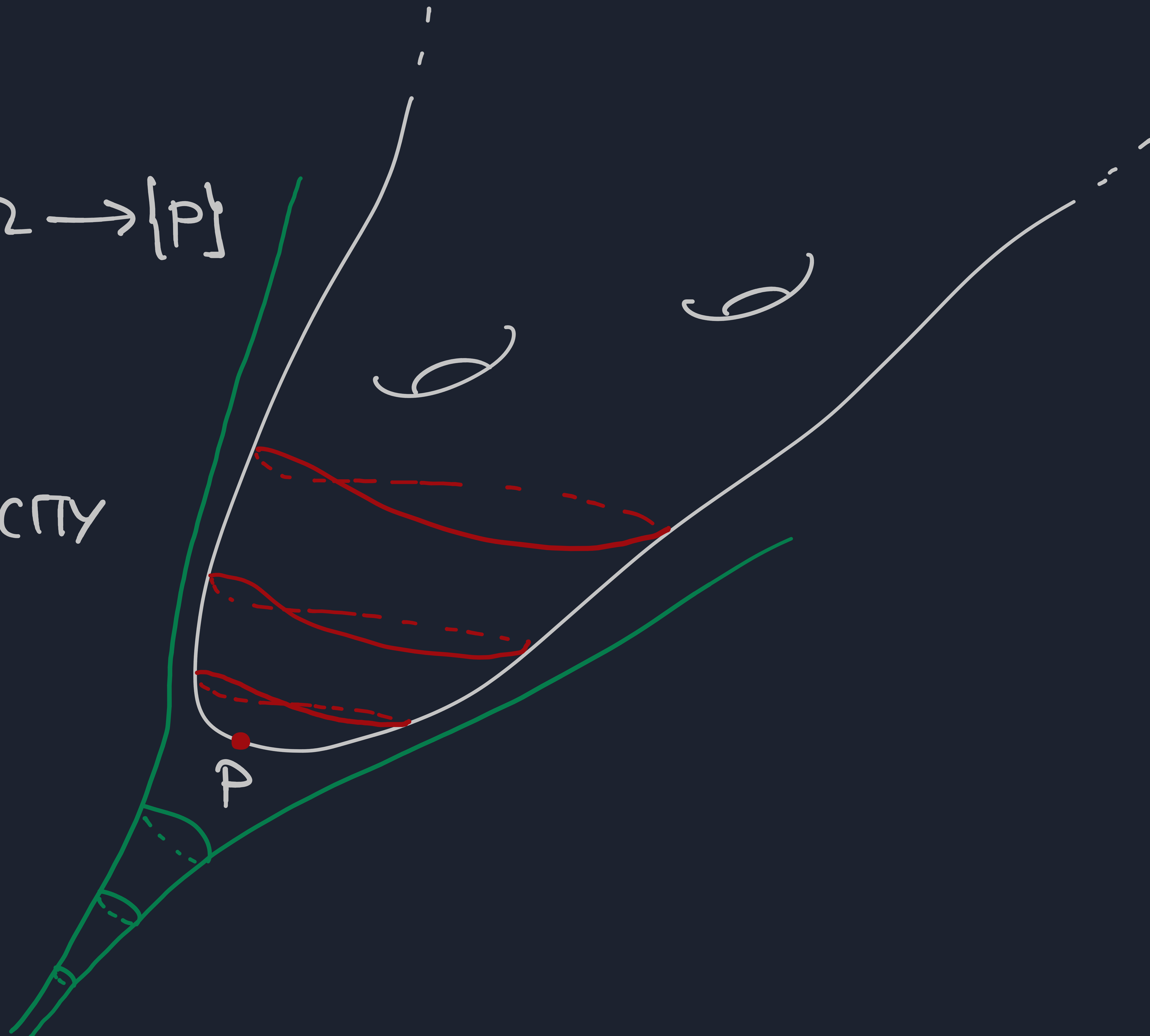
Rmk

Letting $\Omega \to \{p\}$

$\Downarrow$

COLDING MONOTONICITY FORMULAS for GREEN'S FUNCTION

(Acta, 2012)



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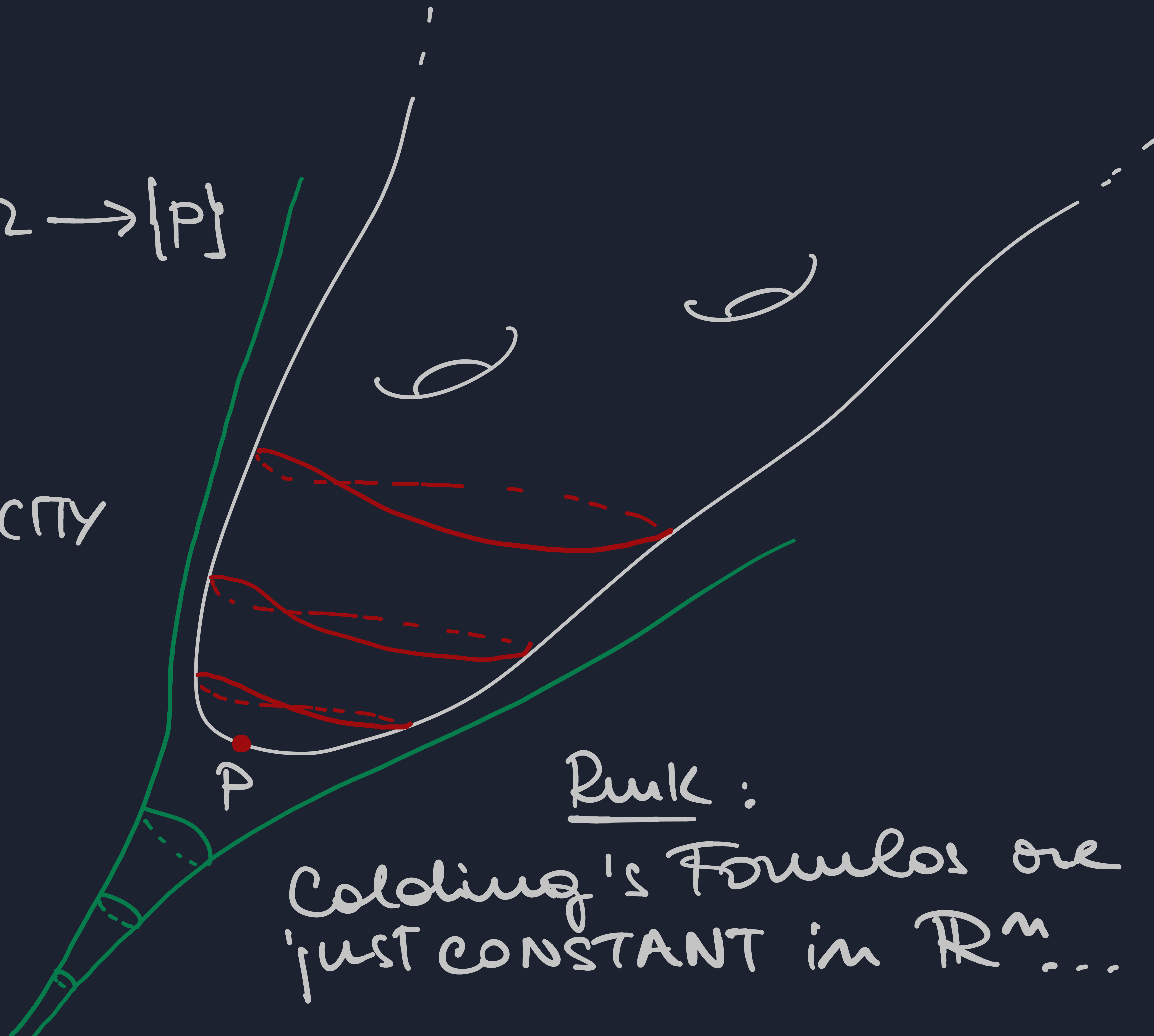


Rmk:

Colding's Formulas are 'just CONSTANT in $\mathbb{R}^n$'...

3) A.F. 3-MANIFOLD WITH $R_g \geq 0$

$$R_g = tr_g(Ric_g) = g^{ij} R_{ij}$$

SCALAR CURVATURE

DEF

- (M^3, g) A.F. $\iff$
  - $M^3 \setminus \underset{cpt.}{K} = \mathbb{R}^3 \setminus \mathbb{B}^3$
  - $g_{ij} - \delta_{ij} = O_2(|x|^{-\tau})$ with $\tau > 1/2$.
- $$m_{ADM}(M^3, g) = \lim_{r \to +\infty} \frac{1}{16\pi} \int_{S_r} \sum_{i,k=1}^{3} \left(\frac{\partial g_{ik}}{\partial x^i} - \frac{\partial g_{ii}}{\partial x^k} \right) \nu^k \, d\sigma$$

MODEL SOLUTION

Space-like Schwarzschild metric (of mass m)

$$g_m = \left(1 + \frac{m}{2|x|}\right)^4 \cdot \delta_{ij}\, dx^i \otimes dx^j$$

$\boxed{\text{POSITIVE MASS THEOREM}}$ (Schoen-Yau 1979)

(M^3, g) Asympt. Flat (A.F.)
Riem. mfld with $R_g \geq 0$

$\Longrightarrow \quad m_{ADM}(M^3, g) \geq 0$

with $=$ iff $(M^3, g) \underset{\text{isom}}{\cong} (\mathbb{R}^3, g_{\mathbb{R}^3})$

$\boxed{\text{DENSITY THEOREM}}$ (after SCHOEN-YAU)

(M^3, g) smooth complete A.F. Riem. mfld with $R_g \geq 0$

$$\boxed{m = m_{ADM}(M^3, g)}$$

$\forall \varepsilon > 0 \quad \exists\, g_\varepsilon$: metric on $\mathbb{R}^3$ s.t.

(i) $R_{g_\varepsilon} \geq 0$ on $\mathbb{R}^3$

(ii) $|m_\varepsilon - m| \leq \varepsilon$, where $m_\varepsilon = m(\mathbb{R}^3, g_\varepsilon)$

(iii) $\exists$ A.F. coord's (x^1, x^2, x^3) s.t.

$$g^{(\varepsilon)}_{ij} = \left(1 + \frac{m_\varepsilon}{2|x|}\right)^4 \delta_{ij} \quad \text{on } \mathbb{R}^3 \setminus B_R .$$

By the DENSITY THM it is sufficient to prove that

$$\boxed{\begin{array}{l} \bullet\ M^3 \simeq \mathbb{R}^3 \text{ with } R_g \geq 0 \\ \bullet\ g = \left(1+\frac{m}{2r}\right)^4 g_{\mathbb{R}^3} \text{ outside a cpt. set} \end{array} \implies m \geq 0}$$

Sketch: Assume by contradiction that $m_{ADM}(M^3, g) < 0$ for some M^3.

Then one can "approximate" M^3 with $(\mathbb{R}^3, g_\varepsilon)$

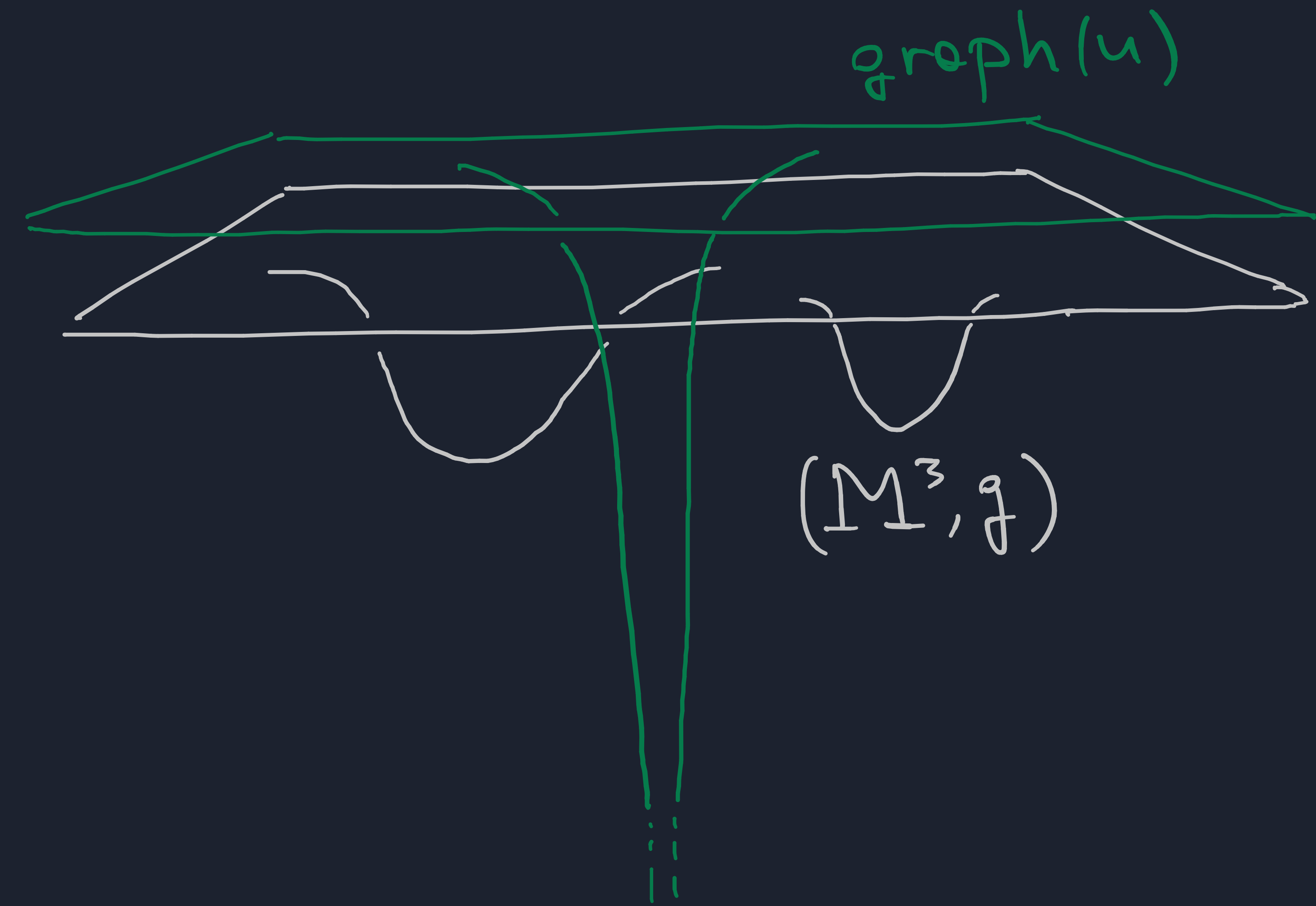
s.t. • $R_{g_\varepsilon} \geq 0$

• $m_\varepsilon \leq m_{ADM} - \varepsilon < 0$ (for suff. small $\varepsilon > 0$)

• g_ε is Schwarz. outside a cpt. set.

<u>proof:</u> Let $o \in M^3$ and let u be the (distributional) solution to:

$$\begin{cases} \Delta u = 4\pi\,\delta_0 & , M^3 \\ u \longrightarrow 1 & , \text{at } \infty \end{cases}$$



proof: Let $o \in M^3$ and let u be the (distributional) solution to:

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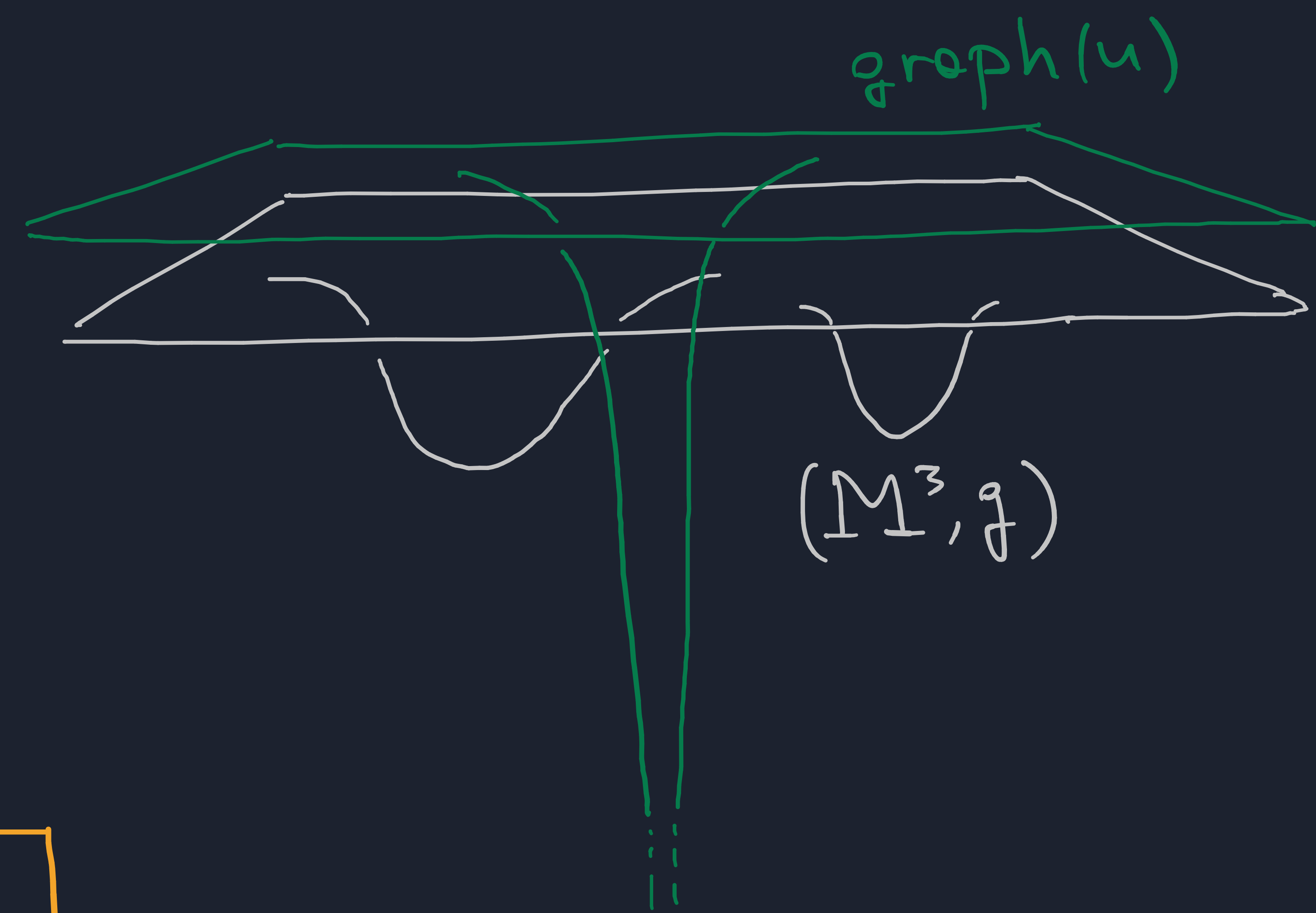

MONOTONICITY FORMULA

$$E(t) \doteq 4\pi t - t^2 \int_{\{u = 1 - 1/t\}} |\nabla u|\, H \, d\sigma + t^3 \int_{\{u = 1 - 1/t\}} |\nabla u|^2 \, d\sigma \,, \quad t \in \mathbb{R}^+$$

E is $\boxed{\text{A.C.}}$ & $\dfrac{dE}{dt} \geq 0$ a.e.

$\boxed{\text{MONOTONICITY FORMULA}} \Longrightarrow \boxed{E(0) \doteq \lim_{t\to 0^+} E(t) \leq \lim_{T\to +\infty} E(T) \doteq E(+\infty)}$

TASK: Compute $E(0)$ & $E(+\infty)$

$\boxed{\text{Asymptotics about } o \in M^3}$

GENERAL FACT

$\boxed{\text{Asymptotics at } \infty}$

SPECIFIC TO OUR ASYMPT.

MONOTONICITY FORMULA $\Longrightarrow$ $\boxed{E(0) \doteq \lim_{t\to 0^+} E(t) \leq \lim_{T\to+\infty} E(T) \doteq E(+\infty)}$

TASK: Compute $E(0)$ & $E(+\infty)$

Asymptotics about ∂M^3

$$\begin{cases} \dfrac{C_1}{r} \leq 1-u \leq \dfrac{C_2}{r} \\ \dfrac{C_3}{r^2} \leq |\nabla u| \leq \dfrac{C_4}{r^2} \\ |\nabla\nabla u| \leq \dfrac{C_5}{r^3} \end{cases} \Longrightarrow \dots \Longrightarrow \boxed{E(0) = 0}$$

COROLLARY

$\forall t \geq 0$: $E(t) \geq 0$. Moreover:

$E(t) = 0 \iff \left(\{u \leq 1 - \frac{1}{t}\}, g\right)$ is a FLAT EUCLIDEAN BALL

Asymptotics at ∞

$$\boxed{\begin{aligned} &u(x) = 1 - \frac{1}{|x|} + \frac{m + \phi\left(\frac{x}{|x|}\right)}{2\,|x|^2} + O_2\left(|x|^{-3+\alpha}\right) \\ &\text{where } 0 < \alpha < 1/2 \quad \& \quad -\Delta_{\mathbb{S}^2}\phi = 2\phi \end{aligned}}$$

$$E(T) = \int_{\{u = 1 - 1/T\}} \left[\frac{1}{1-u} + \frac{\langle D|Du| \mid Du\rangle}{(1-u)^2\,|Du|^2} + \frac{|Du|}{(1-u)^3}\right] \cdot |Du| \, d\sigma$$

$$\Downarrow$$

$$[\ \ldots\ ]$$

$$\Downarrow$$

$$E(+\infty) = 8\pi m$$

proof of the MONOTONICITY FORMULA:

$$E(t) \doteq 4\pi t - t^2 \int_{\Sigma_t} |\nabla u| H \, d\sigma + t^3 \int_{\Sigma_t} |\nabla u|^2 \, d\sigma$$

where $\Sigma_t = \{ u = 1 - 1/t \}$

$$\boxed{\text{CASE : } |\nabla u| \neq 0 \quad \text{EVERYWHERE}}$$

$$\frac{dE}{dt}(t) = 4\pi - 2t \int_{\Sigma_t} |\nabla u| H \, d\sigma + 3t^2 \int_{\Sigma_t} |\nabla u|^2 \, d\sigma$$

$$- t^2 \cdot \frac{d}{dt} \int_{\Sigma_t} |\nabla u| H \, d\sigma + t^3 \cdot \frac{d}{dt} \int_{\Sigma_t} |\nabla u|^2 \, d\sigma$$

- $$\frac{d}{dt}\int_{\Sigma_t} |\nabla u|^2 \, d\sigma = -\frac{1}{t^2}\cdot\int_{\Sigma_t} |\nabla u| H \, d\sigma$$

- $$\frac{d}{dt}\int_{\Sigma_t} |\nabla u| H \, d\sigma = \int_{\Sigma_t}\left[\cancel{\frac{\partial |\nabla u|}{\partial u}\cdot H} + |\nabla u|\frac{\partial H}{\partial u} + \cancel{H^2}\right] d\sigma \cdot \frac{1}{t^2} =$$

$$\frac{\partial H}{\partial u} = -\Delta^{\Sigma_t}\left(\frac{1}{|\nabla u|}\right) - \frac{1}{|\nabla u|}\left[|h|^2 + Ric\left(\frac{\nabla u}{|\nabla u|},\frac{\nabla u}{|\nabla u|}\right)\right]$$

$$= -\frac{1}{t^2}\cdot\int_{\Sigma_t} \left|\frac{\nabla^T|\nabla u|}{|\nabla u|}\right|^2 + |h|^2 + Ric\left(\frac{\nabla u}{|\nabla u|},\frac{\nabla u}{|\nabla u|}\right) d\sigma =$$

$$-\Delta^{\Sigma_t}(\log|\nabla u|) + \left|\frac{\nabla^T|\nabla u|}{|\nabla u|}\right|^2$$

$$R^{\Sigma_t} = R - 2\,Ric(\nu,\nu) + H^2 - |h|^2$$

GAUSS EQUATION

$$= -\frac{1}{t^2}\cdot\int_{\Sigma_t} \left|\frac{\nabla^T|\nabla u|}{|\nabla u|}\right|^2 + \frac{|\mathring{h}|^2}{2} + \frac{R}{2} + \frac{3}{4}H^2 - \frac{R^{\Sigma_t}}{2} \, d\sigma$$

- $$\frac{d}{dt}\int_{\Sigma_t} |\nabla u|^2 \, d\sigma = -\frac{1}{t^2} \cdot \int_{\Sigma_t} |\nabla u| H \, d\sigma$$

- $$\frac{d}{dt}\int_{\Sigma_t} |\nabla u| H \, d\sigma =$$

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Recall : $E(t) := 4\pi t - t^2 \int_{\Sigma_t} |\nabla u| H \, d\sigma + t^3 \int_{\Sigma_t} |\nabla u|^2 \, d\sigma$

$$\frac{dE}{dt} = 4\pi - 2t \int_{\Sigma_t} |\nabla u| H \, d\sigma + 3t^2 \int_{\Sigma_t} |\nabla u|^2$$

$$+ \int_{\Sigma_t} \left| \frac{\nabla^T |\nabla u|}{|\nabla u|} \right|^2 + \frac{|h^\circ|^2}{2} + \frac{R}{2} + \frac{3}{4} H^2 - \frac{R^{\Sigma_t}}{2} \, d\sigma$$

$$- t \int_{\Sigma_t} |\nabla u| H \, d\sigma$$

<u>Recall</u>: $E(t) := 4\pi t - t^2 \int_{\Sigma_t} |\nabla u| H \, d\sigma + t^3 \int_{\Sigma_t} |\nabla u|^2 \, d\sigma$

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$$\frac{dE}{dt} = \overbrace{4\pi}^{\text{GAUSS BONNET}} - 2t \int_{\Sigma_t} |\nabla u| H \, d\sigma + 3t^2 \int_{\Sigma_t} |\nabla u|^2$$

$$+ \int_{\Sigma_t} \underbrace{\left| \frac{\nabla^T |\nabla u|}{|\nabla u|} \right|^2 + \frac{|\mathring{h}|^2}{2} + \frac{R}{2}}_{\text{POSITIVE TERMS}} + \frac{3}{4} H^2 - \underbrace{\frac{R^{\Sigma_t}}{2}}_{\text{GAUSS BONNET}} \, d\sigma$$

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GAUSS BONNET

$$\frac{dE}{dt} = 4\pi - 2t \int_{\Sigma_t} |\nabla u| H \, d\sigma + 3t^2 \int_{\Sigma_t} |\nabla u|^2$$

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POSITIVE TERMS

GAUSS BONNET

$$- t \int_{\Sigma_t} |\nabla u| H \, d\sigma$$

$$= 4\pi - \int_{\Sigma_t} \frac{R^{\Sigma_t}}{2} \, d\sigma + \int_{\Sigma_t} \left| \frac{\nabla^T |\nabla u|}{|\nabla u|} \right|^2 + \frac{|h^\circ|^2}{2} + \frac{R}{2} \, d\sigma +$$

$$+ \frac{3}{4} \int_{\Sigma_t} H^2 - 4 \left| \frac{\nabla u}{1-u} \right| H + 4 \left| \frac{\nabla u}{1-u} \right|^2 d\sigma$$

<u>Recall</u> : $E(t) := 4\pi t - t^2 \int_{\Sigma_t} |\nabla u| H \, d\sigma + t^3 \int_{\Sigma_t} |\nabla u|^2 \, d\sigma$

$$\frac{dE}{dt} = \overset{\text{GAUSS BONNET}}{4\pi} - 2t \int_{\Sigma_t} |\nabla u| H \, d\sigma + 3t^2 \int_{\Sigma_t} |\nabla u|^2$$

$$+ \int_{\Sigma_t} \underbrace{\left|\frac{\nabla^T |\nabla u|}{|\nabla u|}\right|^2 + \frac{|h^\circ|^2}{2} + \frac{R}{2}}_{\text{POSITIVE TERMS}} + \frac{3}{4} H^2 - \underbrace{\frac{R^{\Sigma_t}}{2}}_{\text{GAUSS BONNET}} \, d\sigma$$

$$- t \int_{\Sigma_t} |\nabla u| H \, d\sigma$$

$$= 4\pi - \int_{\Sigma_t} \frac{R^{\Sigma_t}}{2} \, d\sigma + \int_{\Sigma_t} \left|\frac{\nabla^T |\nabla u|}{|\nabla u|}\right|^2 + \frac{|h^\circ|^2}{2} + \frac{R}{2} \, d\sigma +$$

$$+ \frac{3}{4} \int_{\Sigma_t} \left(H - 2 \left| \frac{\nabla u}{1-u} \right| \right)^2 d\sigma \geq 0 \qquad \square$$

if $\nabla u \neq 0$ everywhere

GENERAL CASE

- We prove that $\forall\, s,t \in \mathbb{R}^+$ regular values:

$$\boxed{s \le t \implies E(s) \le E(t)}$$

Sufficient to deduce P.M.T.

- For the $\boxed{\text{A.C.}}$ statement see: BENATTI, FOGAGNOLO, —— "Minkowski Inequality on complete Riem. mfld. with nonnegative Ricci curvature" (2021)

- Wherever $\nabla u \neq 0$, we let:

$$\boxed{X = \frac{\nabla u}{1-u} + \frac{\nabla |\nabla u|}{(1-u)^2} + \frac{|\nabla u|}{(1-u)^3} \nabla u}$$

and observe that:

$$\int_{\Sigma_t} \left\langle X \,\middle|\, \frac{\nabla u}{|\nabla u|} \right\rangle d\sigma = \int_{\Sigma_t} \left[\frac{1}{1-u} + \frac{\langle \nabla |\nabla u| \,|\, \nabla u \rangle}{(1-u)^2 |\nabla u|^2} + \frac{|\nabla u|}{(1-u)^3} \right] |\nabla u| \, d\sigma = E(t)$$

EXTERIOR UNIT NORMAL

Assume $\left(\begin{matrix}\text{for the}\\ \text{last time}\end{matrix}\right)$ NO CRITICAL VALUES in $[1-1/s, 1-1/t]$:

$$E(t) - E(s) = \int_{\{1-1/s < u < 1-1/t\}} \operatorname{div}(X)\, d\mu =$$

COAREA FORMULA

$$= \int_{1-1/s}^{1-1/t} d\tau \int_{\{u=\tau\}} \frac{\operatorname{div}(X)}{|\nabla u|}\, d\sigma =$$

$$= \int_{1-1/s}^{1-1/t} \frac{d\tau}{(1-\tau)^2} \int_{\{u=\tau\}} \left[|\nabla u| - \frac{R^\Sigma}{2} \right] + \left[\left| \frac{\nabla^\Sigma |\nabla u|}{|\nabla u|} \right|^2 + \frac{|\mathring{h}|^2}{2} + \frac{R}{2} \atop \frac{3}{2}\left(H - 2\left|\frac{\nabla u}{1-u}\right| \right)^2 \right] d\sigma \geqslant$$

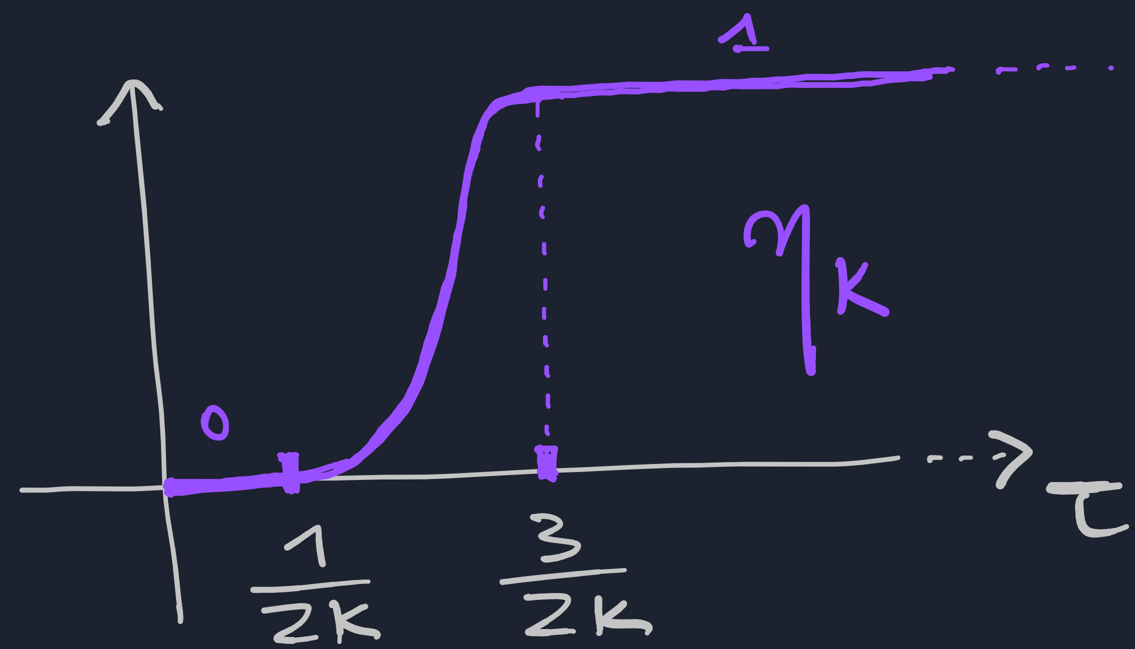
$$\geqslant \int_{1-1/s}^{1-1/t} \frac{d\tau}{(1-\tau)^2} \left[4\pi - \int_{\{u=\tau\}} \frac{R^\Sigma}{2}\, d\sigma \right] \geqslant 0$$

Rmk Under our assumptions every regular level set is connected by the MAXIMUM PRINCIPLE.

PROVIDED ALMOST EVERY LEVEL SET IS CONNECTED

If critical values are present in $[1-1/s, 1-1/t]$:

- $\forall\, k \in \mathbb{N}$ define the smoot cut-off function η_k as in the picture, with $0 \le \dot{\eta}_k(\tau) \le 2K$



$\tau \in \left[\frac{1}{2k}, \frac{3}{2k}\right]$

- Use η_k to define the smooth vector field:

$$X_k = \frac{\nabla u}{1-u} + \eta_k\left(\frac{|\nabla u|}{1-u}\right)\left[\frac{\nabla|\nabla u|}{(1-u)^2} + \frac{|\nabla u|}{(1-u)^3}\nabla u\right]$$

$\boxed{\text{Rmk}}$

$$\nabla \eta_k\left(\frac{|\nabla u|}{1-u}\right) = \dot{\eta}_k\left(\frac{|\nabla u|}{1-u}\right)\cdot\left[\frac{\nabla|\nabla u|}{1-u} + \frac{|\nabla u|}{(1-u)^2}\nabla u\right]$$

$$= \dot{\eta}_k\left(\frac{|\nabla u|}{1-u}\right)\cdot\left[\frac{\nabla|\nabla u|}{(1-u)^2} + \frac{|\nabla u|}{(1-u)^3}\nabla u\right](1-u)$$

Compute the divergence of X_k:

$$\operatorname{div}(X) = \frac{|\nabla u|}{(1-u)^2}\left[|\nabla u| + \frac{3|\nabla u|^2}{(1-u)^2} + \frac{3\langle \nabla|\nabla u| \,|\, \nabla u\rangle}{(1-u)\,|\nabla u|} + \frac{|\nabla\nabla u|^2 - |\nabla|\nabla u||^2}{|\nabla u|^2} + \frac{\operatorname{Ric}(\nabla u, \nabla u)}{|\nabla u|^2}\right]$$

without cut-off

with cut-off

$$\operatorname{div}(X_k) = \frac{|\nabla u|}{(1-u)^2}\cdot\left\{ |\nabla u| + \eta_k\left(\frac{|\nabla u|}{1-u}\right)\left[\frac{3|\nabla u|^2}{(1-u)^2} + \frac{|\nabla\nabla u|^2 - |\nabla|\nabla u||^2}{|\nabla u|^2}\right]\right\}$$

$$+ \frac{|\nabla u|}{(1-u)^2}\cdot\left\{ \eta_k\left(\frac{|\nabla u|}{1-u}\right)\left[\frac{3\langle \nabla|\nabla u| \,|\, \nabla u\rangle}{(1-u)\,|\nabla u|} + \operatorname{Ric}\left(\frac{\nabla u}{|\nabla u|}, \frac{\nabla u}{|\nabla u|}\right)\right]\right\}$$

$$+ \frac{|\nabla u|^2}{(1-u)^3}\,\dot{\eta}_k\left(\frac{|\nabla u|}{1-u}\right)\left|\frac{\nabla u}{1-u} - \frac{\nabla|\nabla u|}{|\nabla u|}\right|^2$$

Compute the divergence of X_k:

$$\operatorname{div}(X_k) \geq \frac{|\nabla u|^2}{(1-u)^2} + \overbrace{\frac{|\nabla u|}{(1-u)^2}\cdot\left\{\eta_k\left(\frac{|\nabla u|}{1-u}\right)\left[\frac{3|\nabla u|^2}{(1-u)^2} + \frac{|\nabla\nabla u|^2 - |\nabla|\nabla u||^2}{|\nabla u|^2}\right]\right\}}^{P_k}$$

$|\nabla u| +$

$$+ \overbrace{\frac{|\nabla u|}{(1-u)^2}\cdot\left\{\eta_k\left(\frac{|\nabla u|}{1-u}\right)\left[\frac{3\langle\nabla|\nabla u|, \nabla u\rangle}{(1-u)\,|\nabla u|} + \operatorname{Ric}\left(\frac{\nabla u}{|\nabla u|}, \frac{\nabla u}{|\nabla u|}\right)\right]\right\}}^{D_k}$$

$$\cancel{+ \frac{|\nabla u|^2}{(1-u)^3}\,\dot\eta_k\left(\frac{|\nabla u|}{1-u}\right)\left|\frac{\nabla u}{1-u} - \frac{\nabla|\nabla u|}{|\nabla u|}\right|^2}$$

Hence:

$$E(t) - E(s) \geq \int_{\{1-1/s<u<1-1/t\}} \frac{|\nabla u|^2}{(1-u)^2}\,d\mu$$

$$+ \int_{\{1-1/s<u<1-1/t\}} P_k\,d\mu + \int_{\{1-1/s<u<1-1/t\}} D_k\,d\mu$$

• Observe that:

$$|D_k| = \left| \frac{|\nabla u|}{(1-u)^2} \cdot \eta_k\left(\frac{|\nabla u|}{1-u}\right) \cdot \left[\frac{3\langle \nabla |\nabla u| \, | \nabla u \rangle}{(1-u)\,|\nabla u|} + \mathrm{Ric}\left(\frac{\nabla u}{|\nabla u|}, \frac{\nabla u}{|\nabla u|}\right) \right] \right|$$

$$\leq \frac{|\nabla u|}{(1-u)^2} \left[\frac{3\,|\nabla\nabla u|}{1-u} + |\mathrm{Ric}| \right] \cdot \mathbb{1}_{M \setminus \mathrm{Crit}(u)} \in L^1_{loc}$$

By the DOMINATED CONV. THM:

$$\lim_{k \to +\infty} \int_{\{1-1/s < u < 1-1/t\}} D_k \, d\mu = \int_{\{1-1/s < u < 1-1/t\} \setminus \mathrm{Crit}(u)} \frac{|\nabla u|}{(1-u)^2} \left[\frac{3\langle \nabla |\nabla u| \, | \nabla u \rangle}{(1-u)\,|\nabla u|} + \mathrm{Ric}\left(\frac{\nabla u}{|\nabla u|}, \frac{\nabla u}{|\nabla u|}\right) \right] d\mu$$

$$\lim_{k \to +\infty} \eta_k\left(\frac{|\nabla u|}{1-u}\right)(x) = \mathbb{1}_{M \setminus \mathrm{Crit}(u)}(x)$$

• Observe that $\{P_k\}_{k\in\mathbb{N}}$ is a nondecreasing sequence of nonnegative functions and :

$$\lim_{k\to+\infty} P_k(x) = \frac{|\nabla u|}{(1-u)^2}(x)\left\{ \frac{3|\nabla u|^2}{(1-u)^2}(x) + \frac{|\nabla\nabla u|^2 - |\nabla|\nabla u||^2}{|\nabla u|^2}(x)\right\}\cdot \mathbb{1}_{M\setminus \mathrm{Crit}(u)}(x)$$

By the MONOTONE CONVERGENCE THM :

$$\lim_{k\to+\infty} \int_{\{1-1/s<u<1-1/t\}} P_k \, d\mu = \int_{\{1-1/s<u<1-1/t\}\setminus \mathrm{Crit}(u)} \frac{|\nabla u|}{(1-u)^2}\left[\frac{3|\nabla u|^2}{(1-u)^2} + \frac{|\nabla\nabla u|^2 - |\nabla|\nabla u||^2}{|\nabla u|^2}\right] d\mu$$

$\boxed{\text{CONCLUSION}}$

$$E(t) - E(s) \geqslant \int_{\{1-1/s < u < 1-1/t\} \setminus \mathrm{Crit}(u)} \mathrm{div}(X)\, d\mu =$$

$$= \int_{1-1/s}^{1-1/t} d\tau \int_{\{u=\tau\} \setminus \mathrm{Crit}(u)} \frac{\mathrm{div}(X)}{|\nabla u|}\, d\sigma =$$

$\mathcal{N}(u) = u(\mathrm{Crit}(u))$
CRITICAL VALUES:
$\mathcal{N}(u)$ is NEGLIGIBLE
By Sard's Thm.

$$= \int_{[1-1/s,\, 1-1/t] \setminus \mathcal{N}(u)} d\tau \int_{\{u=\tau\}} \frac{\mathrm{div}(X)}{|\nabla u|}\, d\sigma =$$

$$= \int_{[1-1/s,\, 1-1/t] \setminus \mathcal{N}(u)} \frac{d\tau}{(1-\tau)^2} \int_{\{u=\tau\}} \left[|\nabla u| - \frac{R^\Sigma}{2} \right] + \left[\left| \frac{\nabla^\Sigma |\nabla u|}{|\nabla u|} \right|^2 + \frac{|\mathring{h}|^2}{2} + \frac{R}{2} \atop \frac{3}{2}\left(H - 2\left|\frac{\nabla u}{1-u}\right| \right)^2 \right] d\sigma$$

$\square$

4) NONLINEAR POTENTIAL THEORY

$$\begin{cases} \Delta_p u = 0 & M^3 \\ u = 0 & \partial M^3 \\ u \longrightarrow 1 & \text{at } \infty \end{cases}$$

$$E_p(t) := 4\pi t - \frac{t^{\frac{2}{p-1}}}{c_p} \int_{\left\{u = 1 - \frac{c_p\left(\frac{p-1}{3-p}\right)}{t^{\frac{3-p}{p-1}}}\right\}} |\nabla u| \, H \, d\sigma + \frac{t^{\frac{5-p}{p-1}}}{c_p^2} \int_{\left\{u = 1 - \frac{c_p\left(\frac{p-1}{3-p}\right)}{t^{\frac{3-p}{p-1}}}\right\}} |\nabla u|^2 \, d\sigma$$

where

$$c_p := \left(\frac{\mathrm{Cap}_p(\partial M)}{4\pi} \right)^{\frac{1}{p-1}}$$

$|\partial M^*|$ area of the (strictly) outward minimizing envelope of ∂M.

THEOREM

$$\boxed{\mathrm{Cap}_p(\partial \Omega) \longrightarrow |\partial M^*| \quad \text{as } p \to 1^+} \implies \lim_{p \to 1^+} c_p^{p-1} = \frac{|\partial M^*|}{4\pi}$$

- Assume ∂M is OUTERMOST MINIMAL SURF.

and let $t_p = \left[c_p \left(\frac{p-1}{3-p} \right) \right]^{\frac{p-1}{3-p}}$.

$$E_p(t_p) = 4\pi t_p + 0 + \frac{t_p^{\frac{5-p}{p-1}}}{c_p^2} \int_{\partial M} |\nabla u|^2 \, d\sigma \geq 4\pi t_p$$

$\wedge\!\!\!\!/\!/$ ← $\boxed{\text{MONOTONICITY}}$

$E_p(+\infty) = 8\pi m$ ← $\boxed{\text{ASYMPTOTICS}}$

Hence:

$$2m \geq t_p = \left[c_p \left(\frac{p-1}{3-p} \right) \right]^{\frac{p-1}{3-p}} =$$

$$= \left(\frac{\mathrm{Cap}_p(\partial M)}{4\pi} \right)^{\frac{1}{3-p}} \cdot \left(\frac{p-1}{3-p} \right)^{\frac{p-1}{3-p}} \longrightarrow \sqrt{\frac{|\partial M^*|}{4\pi}}$$

as $p \to 1^+$

OUTERMOST MINIMAL SURF. are OUTWARD MINIMIZING

$$\sqrt{\frac{|\partial M|}{4\pi}} \; = \; \sqrt{\frac{|\partial M^*|}{4\pi}}$$

RIEMANNIAN PENROSE INEQUALITY

(M^3, g) A.F. Riemannian manifold with $R_g \geq 0$.

Assume $\partial M \neq \phi$ is:

- CONNECTED
- OUTERMOST MINIMAL SURFACE

($\nexists$ other minimal surfaces enclosing ∂M)

$$\Longrightarrow \quad m_{ADM}(M^3, g) \geq \sqrt{\frac{|\partial M|}{16\pi}}$$

LINK WITH GEROCH MONOTONICITY

monotonicity of the HAWKING MASS along the INVERSE MEAN CURVATURE FLOW

Set $$\boxed{u = 1 - c_p\left(\frac{p-1}{3-p}\right) e^{-\left(\frac{3-p}{p-1}\right)\phi/2}}$$

$$\rightsquigarrow \quad \nabla u = \left(\frac{c_p}{2}\right) \cdot e^{-\left(\frac{3-p}{p-1}\right)\phi/2}\, \nabla\phi$$

$$\rightsquigarrow \quad |\nabla u|^{p-2}\nabla u = \left(\frac{c_p}{2}\right)^{p-1} \cdot e^{-\left(\frac{3-p}{2}\right)\phi}\, |\nabla\phi|^{p-2}\nabla\phi$$

$$\rightsquigarrow \quad 0 = \mathrm{div}\left(|\nabla u|^{p-2}\nabla u\right) =$$

$$= \left(\frac{c_p}{2}\right)^{p-1} e^{-\frac{3-p}{2}\phi}\left(\Delta_p\phi - \left(\frac{3-p}{2}\right)|\nabla\phi|^p\right)$$

Hence: $$\boxed{\Delta_p\phi = \left(\frac{3-p}{2}\right)|\nabla\phi|^p}$$

Setting $t = e^{\tau/2}$ and using the ϕ-variable,

$$\Delta_p \phi = \left(\frac{3-p}{2}\right) |\nabla \phi|^p$$

the monotonic quantities become:

$$M_p(\tau) = e^{\tau/2}\left(4\pi + \int_{\{\phi=\tau\}} \frac{|\nabla\phi|}{2}\left(\frac{|\nabla\phi|}{2} - H\right) d\sigma\right)$$

Setting $t = e^{\tau/2}$ and using the ϕ-variable,

$$\Delta_p \phi = \left(\frac{3-p}{2}\right) |\nabla \phi|^p$$

$$\Big\downarrow \text{ as } p \to 1^+$$

$$H = \operatorname{div}\left(\frac{\nabla \phi}{|\nabla \phi|}\right) = \Delta_1 \phi = 1 \cdot |\nabla \phi|^1 = |\nabla \phi| \qquad \boxed{\text{IMCF}}$$

the monotonic quantities become:

$$\mathcal{M}_p(\tau) = e^{\tau/2} \left(4\pi + \int_{\{\phi = \tau\}} \frac{|\nabla \phi|}{2} \left(\frac{|\nabla \phi|}{2} - H \right) d\sigma \right)$$

$$\Big\downarrow \text{ as } p \to 1^+ \qquad \boxed{\text{HAWKING MASS}}$$

$$\mathcal{M}_1(\tau) = e^{\tau/2} \left(4\pi - \int_{\{\phi = \tau\}} \frac{H^2}{4} \, d\sigma \right) =$$

$$= 4\pi \sqrt{\frac{|\Sigma_\tau|}{|\Sigma_0|}} \left(1 - \frac{1}{16\pi} \int_{\Sigma_\tau} H^2 \, d\sigma \right) = \frac{16 \pi^{3/2}}{\sqrt{|\partial M|}} \cdot m_{HAW}(\Sigma_\tau)$$

5) NONLINEAR POTENTIAL THEORY AND RICCI LOWER BOUNDS

(M, g) complete non cpt. Riem. mfld

- $\mathrm{Ric}_g \geq 0$
- Euclidean volume growth

$0 < \mathrm{AVR}(g) \leq 1$

$\boxed{\text{THEOREM}}$ (Benatti, Fogagnolo, —— 2021)

$$\left[|\mathbb{S}^{n-1}| \, \mathrm{AVR}(g) \right]^{\frac{1}{n-1}} \leq \frac{\int_{\partial\Omega} \frac{H}{n-1} \, d\sigma}{|\partial\Omega^*|^{\frac{n-2}{n-1}}}$$