Multi-Bubble Isoperimetric Problems - Old and New

Emanuel Milman
Technion - Israel Institute of Technology
and Oden Institute at UT Austin

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joint work (in progress) with Joe Neeman (UT Austin)

The Classical Isoperimetric Inequality

"Among all sets in Euclidean space $\mathbb{R}^n$ having a given volume, Euclidean balls minimize surface area."

$$V(\Omega) = V(\text{Ball}) \Rightarrow A(\Omega) \geq A(\text{Ball}).$$

$\Omega \in \mathcal{B}(\mathbb{R}^n)$, $V = \text{Leb}^n$, $A = $ Surface Area.

What is Surface Area? Various (non-equivalent) definitions:

- If $\partial\Omega$ smooth, $\int_{\partial\Omega} d\mathrm{Vol}_{\partial\Omega}$.
- Hausdorff measure $\mathcal{H}^{n-1}(\partial\Omega)$.
- Minkowski exterior boundary measure: $V^+(\Omega) = \liminf_{\epsilon\to 0+} \frac{V(\Omega_\epsilon \setminus \Omega)}{\epsilon}$, $\Omega_\epsilon := \{y \in \mathbb{R}^n ; d(y,\Omega) < \epsilon\}$.
- De Giorgi Perimeter $P(\Omega) = \mathcal{H}^{n-1}(\partial^*\Omega) = \|1_\Omega\|_{BV} = \|\nabla 1_\Omega\|_{TV} = \sup\left\{\int_\Omega \nabla \cdot X ; X \in C_c^\infty(\mathbb{R}^n; T\mathbb{R}^n) , |X| \leq 1\right\}$.
  Stronger than rest, l.s.c., invariant under null-set modifications.

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Isoperimetric Inequalities in Metric-Measure setting

Classical isoperimetric inequality is on $\mathbb{R}^n = (\mathbb{R}^n, |\cdot|, \text{Leb}^n)$.
Study in weighted-manifold setting $(M^n, g, \mu = \Psi(x) d\text{Vol}_g)$, $\Psi > 0$.

Weighted Volume and Area:

- $V(\Omega) = \mu(\Omega) = \int_\Omega \Psi(x) d\text{Vol}_g$.
- $A(\Omega) = P_\Psi(\Omega) = \int_{\partial^* \Omega} \Psi(x) d\mathcal{H}^{n-1}(x)$.

Examples:

1. $\mathbb{S}^n = (\mathbb{S}^n, g_{\text{can}}, \lambda_{\mathbb{S}^n} = \frac{\text{Vol}_{\mathbb{S}^n}}{\text{Vol}(\mathbb{S}^n)})$ - P. Lévy, Schmidt 20-30's: geodesic balls are isoperimetric minimizers.
2. $\mathbb{G}^n = (\mathbb{R}^n, |\cdot|, \gamma^n = \frac{1}{(2\pi)^{n/2}} e^{-\frac{|x|^2}{2}} dx)$ - Sudakov–Tsirelson, Borell '75: half-spaces are isoperimetric minimizers.

Relation (Maxwell, Poincaré, Borel): $(\pi_{\mathbb{R}^n})_*(\lambda_{\sqrt{N}\mathbb{S}^N}) \to_{N \to \infty} \gamma^n$.

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Isoperimetric Inequalities for Clusters

Cluster $\Omega = (\Omega_1, \ldots, \Omega_q)$ is a partition $M = \Omega_1 \cup \ldots \cup \Omega_q$ (up to null-sets)
Given $V(\Omega) = (V(\Omega_1) \ldots V(\Omega_q))$ minimize $A(\Omega) = \frac{1}{2} \sum_{i=1}^{q} A(\Omega_i) = \sum_{i<j} A_{ij}$.

Previous examples: $q = 2$ ($\Omega_1 = U, \Omega_2 = M \setminus U$), "Single Bubble".
Would like to study $q \geq 3$, "Multi Bubble" case.
Case $q = 3$ is called "Double Bubble" $(\Omega_1, \Omega_2, M \setminus (\Omega_1 \cup \Omega_2))$.

- $\mathbb{R}^n$ - Theorem: for all $V(\Omega) = (v_1, v_2, \infty)$, standard double bubble (3 spherical caps meeting at $120°$ along $(n-2)$-dim sphere) minimizes total surface area:

  $\mathbb{R}^2$ - F. Morgan's "SMALL" undergraduate group (Foisy–Alfaro–Brock–Hodges–Zimba) '93.

  $\mathbb{R}^3$ - Hass–Hutchings–Schlafly '95 $v_1 = v_2$, Hutchings–Morgan–Ritoré–Ros '00.

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Isoperimetric Double-Bubble Conjectures

$q = 3$ regions in dimension $n \geq 2$:

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   $\mathbb{S}^2$ - Proved by Masters '96.
   $\mathbb{S}^3$ - Cotton–Freeman '02, Corneli–Hoffman-HLLMS '07, partial.
   $\mathbb{S}^n$ - Corneli–Corwin–Hoffman-HSADLVX '08, if $\left|v_i - \frac{1}{3}\right| \leq 0.04$.

2. $\mathbb{G}^n$ - Double-Bubble Conjecture: for all $V(\Omega) = (v_1, v_2, v_3)$, standard "tripod" / "Y" (3 half-hyperplanes meeting at $120°$ along $(n-2)$-dim plane) minimizes total (Gaussian) surface area.

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   Interaction between $\mathbb{G}$ and $\mathbb{S}$:
   $\mathbb{G}^2 \Rightarrow \mathbb{S}^N \;\; \forall N \gg 1 \Rightarrow \mathbb{S}^n \;\; \forall n \geq 2 \Rightarrow \mathbb{G}^n \;\; \forall n \geq 2$ by projection.

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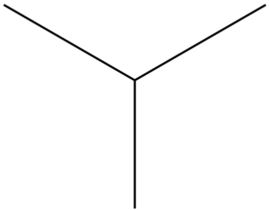
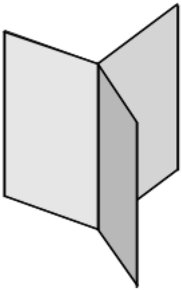
Y cone





Isoperimetric Multi-Bubble Conjectures

Higher-order cluster $\Omega = (\Omega_1, \ldots, \Omega_q)$.
There's no reasonable conjecture when $q \gg n$:

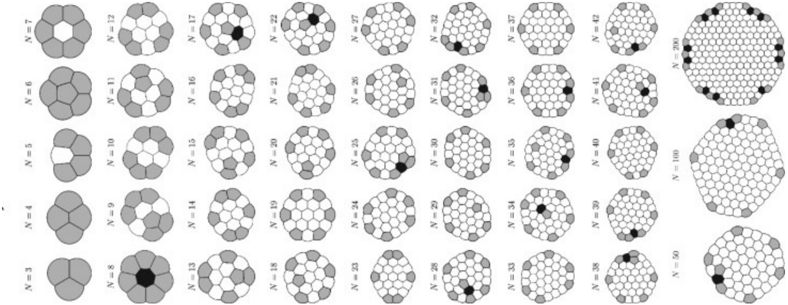


Image from Cox, Graner, et al.

Multi-Bubble Conjecture on $\mathbb{R}^n$ (J. Sullivan '95): If $q - 1 \leq n + 1$, for all $V(\Omega) = (v_1, \ldots, v_{q-1}, \infty)$, the minimizer is a standard $q - 1$ bubble:

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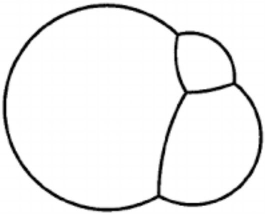

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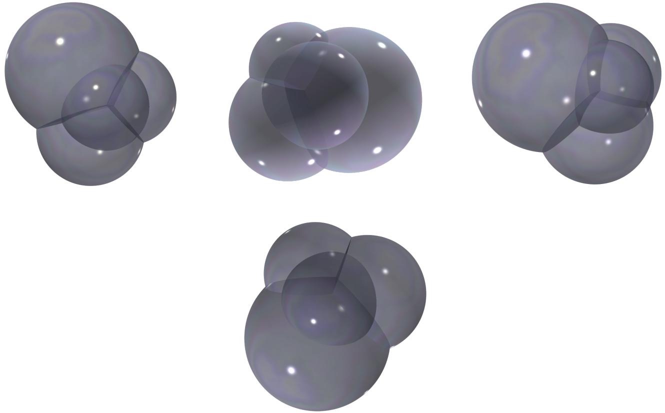

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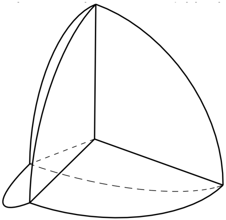

Montesinos Amilibia '01 - standard bubbles exist and are uniquely determined (up to isometries) for all prescribed volumes, for all $q-1 \leq n+1$.

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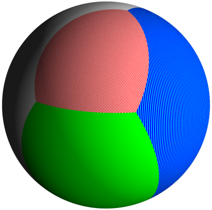

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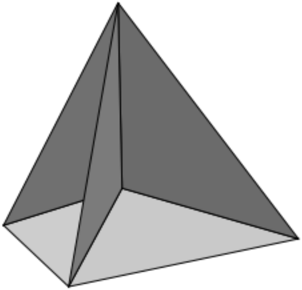

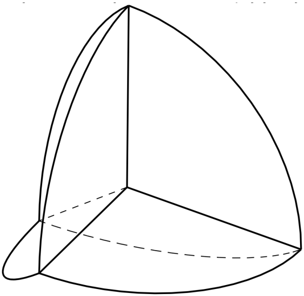

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$q = 2$ corresponds to the classical isoperimetric inqs.
$q = 3$ is the double-bubble theorem ($\mathbb{R}^n$) / conjecture ($\mathbb{S}^n$ / $\mathbb{G}^n$, $n \geq 3$).
$q = 4$ and $n = 2$ in $\mathbb{R}^n$ (planar triple-bubble) proved by Wichiramala '04.
Not aware of any other results when $q \geq 4$ prior to 2018.

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Gaussian Double/Multi-Bubble Thm (M.–Neeman '18)

For all $n \geq 2$ and $2 \leq q \leq n+1$, the Multi-Bubble Conjecture on $\mathbb{G}^n$ is true: "a standard simplicial q-cluster is a Gaussian minimizer".

Gaussian Double/Multi-Bubble Uniqueness (M.–Neeman '18)

For all $n \geq 2$ and $2 \leq q \leq n+1$, simplicial q-clusters are the *unique* minimizers of Gaussian perimeter, up to null-sets.

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Multi-Bubble Conjecture on $\mathbb{S}^n$: If $q-1 \leq n+1$, for all $V(\Omega) = (v_1, \ldots, v_q)$, the minimizer is a standard $q-1$ bubble.

Equal volume case?

1-2-3-4-5-Bubble Thm on $\mathbb{R}^n$ / $\mathbb{S}^n$ (M.–Neeman '22)

For all $n \geq 2$ and $2 \leq q \leq \min(6, n+1)$, the Multi-Bubble Conjecture on $\mathbb{R}^n$ / $\mathbb{S}^n$ is true: "A standard $q-1$ bubble is an isoperimetric minimizer". In other words, Double-Bubble ($n \geq 2$), Triple-Bubble ($n \geq 3$), Quadruple-Bubble ($n \geq 4$), Quintuple-Bubble ($n \geq 5$).

Additional partial results valid for all $q \leq n+1$ later on.

Multi-Bubble Uniqueness on $\mathbb{R}^n$ / $\mathbb{S}^n$ (M.–Neeman '22)

Uniqueness (up to null-sets) on $\mathbb{S}^n$ for $2 \leq q \leq \min(6, n+1)$.
Uniqueness (up to null-sets) on $\mathbb{R}^n$ for $2 \leq q \leq \min(5, n+1)$.

Q: Why is $\mathbb{S}^n$ case harder than $\mathbb{G}^n$? And $\mathbb{R}^n$ case even more so?
A1: $\mathbb{S}^N \Rightarrow \mathbb{G}^n$ by projection; $\mathbb{S}^n \Rightarrow \mathbb{R}^n$ by scale-invariance and

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Uniqueness (up to null-sets) on $\mathbb{S}^n$ for $2 \leq q \leq \min(6, n+1)$.
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Q: Why is $\mathbb{S}^n$ case harder than $\mathbb{G}^n$? And $\mathbb{R}^n$ case even more so?
A1: $\mathbb{S}^N \Rightarrow \mathbb{G}^n$ by projection; $\mathbb{S}^n \Rightarrow \mathbb{R}^n$ by scale-invariance and shrinking to a point, but uniqueness is lost in both cases.

Isoperimetric Multi-Bubble Results - New

Multi-Bubble Conjecture on $\mathbb{R}^n$ (J. Sullivan '95): If $q-1 \leq n+1$, for all $V(\Omega) = (v_1, \ldots, v_{q-1}, \infty)$, the minimizer is a standard $q-1$ bubble.

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For all $n \geq 2$ and $2 \leq q \leq \min(6, n+1)$, the Multi-Bubble Conjecture on $\mathbb{R}^n$ / $\mathbb{S}^n$ is true: "A standard $q-1$ bubble is an isoperimetric minimizer". In other words, Double-Bubble ($n \geq 2$), Triple-Bubble ($n \geq 3$), Quadruple-Bubble ($n \geq 4$), Quintuple-Bubble ($n \geq 5$).

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Tools in Isoperimetric Problems

Single Bubble ($q = 2$):

0. $\mathbb{R}^n$ - symmetrization, Brunn–Minkowski, L^2, heat-flow, PDE, Localization, Optimal-Transport, Combinatorial, GMT.
1. $\mathbb{S}^n$ - symmetrization, GMT, Localization.
2. $\mathbb{G}^n$ - Projection of $\mathbb{S}^N$, symmetrization (Ehrhard), Brunn-Minkowski (Borell), Localization, heat-flow, GMT.

Double-Bubble ($q = 3$):

- Geometric Measure Theory (De Giorgi, Federer, Almgren, ...) - existence and regularity of isoperimetric minimizers.
- Symmetrization (White, Hutchings).
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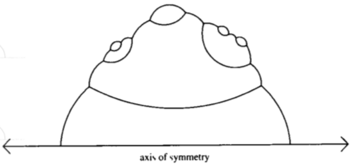

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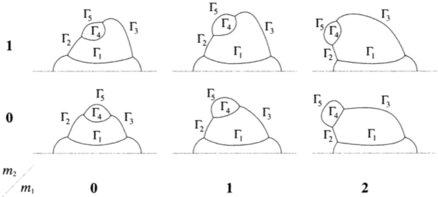

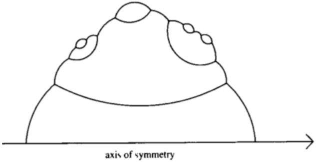

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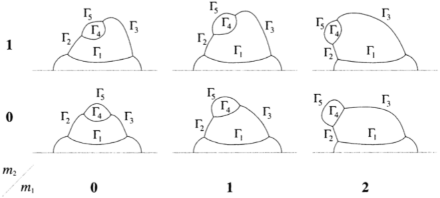

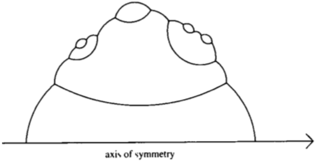

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Starting Point - Geometric Measure Theory

On smooth $(M^n, g, \mu^n = e^{-W} dvol)$, finite volume, GMT guarantees:

- Minimizing $\Omega = (\Omega_1, \ldots, \Omega_q)$ exists (Almgren: also on $\mathbb{R}^n$).
- Interfaces: $\Sigma_{ij} := \partial^* \Omega_i \cap \partial^* \Omega_j$.
- Almgren 70's: Σ_{ij} are C^∞ embedded mnflds w/ good properties. Great books on clusters by F. Morgan and F. Maggi.
- Test against competitors by flowing along vector-field. If $X \in C_c^\infty(M^n; TM^n)$, $\frac{d}{dt} F_t = X \circ F_t$ diffeomorphism, $\Omega_t = F_t(\Omega)$. $V = V(\Omega_t)$, $A = A(\Omega_t)$, $\delta_X^k V = (\frac{d}{dt})^k|_{t=0} V(\Omega_t)$, $\delta_X^k A = (\frac{d}{dt})^k|_{t=0} A(\Omega_t)$.
- Since Ω (globally) minimizes area under volume constraint, there are Lagrange multipliers $\lambda \in E^{(q-1)} = \{ v \in \mathbb{R}^q \; ; \; \sum_{i=1}^q v_i = 0 \}$, s.t.:

  Ω is "stationary" (critical point) $\quad \delta_X^1 A - \langle \lambda, \delta_X^1 V \rangle = 0.$

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- Since the first-variation of (weighted) area is (weighted) mean-curvature, then $H_{\Sigma_{ij}, \mu} = \lambda_i - \lambda_j$ is constant on Σ_{ij}.
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Proof: Step 1 – Minimizer has Trivial Curvature

Stability: $\delta^1_X V = 0 \;\Rightarrow\; 0 \leq Q(X) := \delta^2_X A - \left\langle \lambda, \delta^2_X V \right\rangle$ (the "index-form").

Idea: use stability for vector-fields X_α chosen from an appropriate family, so that $\delta^1_{X_\alpha} V = 0$ and $\mathbb{E}_\alpha Q(X_\alpha) \leq 0$. Read off information on II.

On $\mathbb{G}^n$: $q \leq n+1 \;\Rightarrow\;$ minimizer is flat $\mathrm{II} = 0$.
Maximal case $q = n+1$: separate argument, $Q(\text{Translations}) \leq 0$.

On $\mathbb{S}^n/\mathbb{R}^n$: $q \leq n+1 \;\Rightarrow\;$ minimizer is spherical $\mathrm{II}_0 = \mathrm{II} - \frac{H}{n-1}\mathrm{Id} = 0$.
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This is harder on $\mathbb{S}^n/\mathbb{R}^n$ since $H_{ij} = \lambda_i - \lambda_j$ is unknown, and we need to combine several fields & discover integration by parts formulas.

Step 1 is the critical step – before which we were completely stuck.

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Proof: Steps 2 & 3 – Minimizer is Voronoi Cluster

On $\mathbb{G}^n$: These steps not needed; jump to Step 4!

On $\mathbb{S}^n/\mathbb{R}^n$: $q \leq n+1 \;\Rightarrow\;$ minimizer is spherical Voronoi cluster:
There exist $\{\mathfrak{c}_i\}_{i=1,\ldots,q} \subset \mathbb{R}^{n+1}/\mathbb{R}^n$ and $\{\kappa_i\}_{i=1,\ldots,q} \subset \mathbb{R}$ so that:

1. For every $\Sigma_{ij} \neq \emptyset$, Σ_{ij} lies on a (generalized) geodesic sphere S_{ij} with quasi-center $\mathfrak{c}_{ij} = \mathfrak{c}_i - \mathfrak{c}_j$ and curvature $\kappa_{ij} = \kappa_i - \kappa_j$.
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Step 2 involves simplicial homology of $\{\Omega_i\}_{i=1,\ldots,q}$, Convex Geometry.
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Proof: Steps 2 & 3 – Minimizer is Voronoi Cluster

On $\mathbb{G}^n$: These steps not needed; jump to Step 4!

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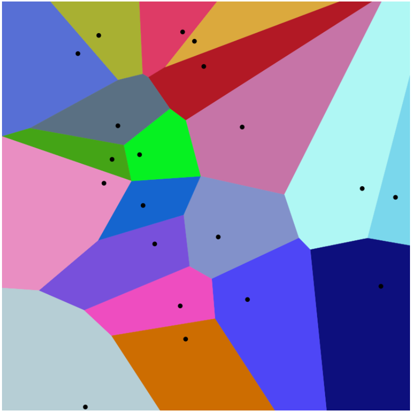

Euclidean Voronoi Cells:

$$\Omega_i = \{x : \arg\min_j |x - x_j|^2 = i\}$$

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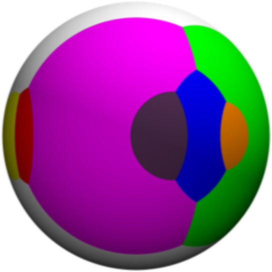

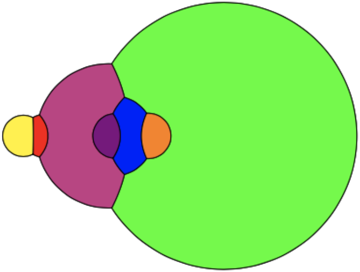

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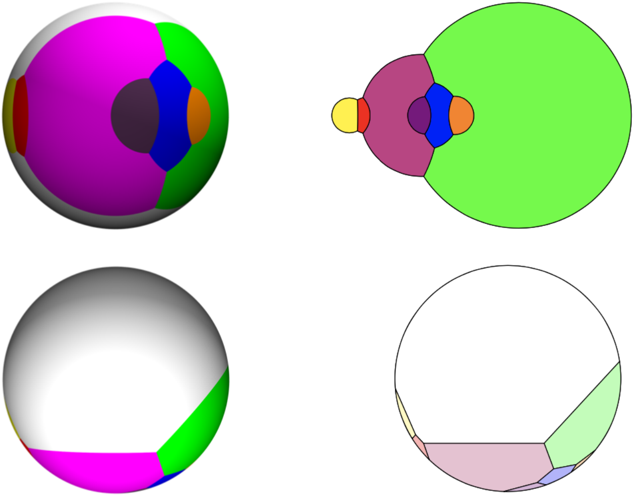

Proof: Step 4 – Need Global Information

At this point, we know that our cluster is spherical / flat Voronoi.
We are almost done! **Fact:** class of Voronoi clusters with $\Sigma_{ij} \neq \emptyset \ \forall i < j$ coincides with the class of conjectured minimizers.

We now need to incorporate a global argument, as local arguments (e.g. stability) will never be enough to exclude configurations like:

Typical GMT argument: if cluster non-rigid, move bubbles until they touch, forming an illegal singularity for an isoperimetric cluster.

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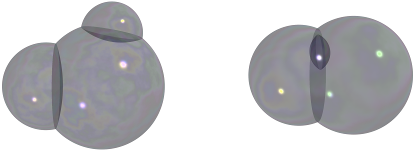
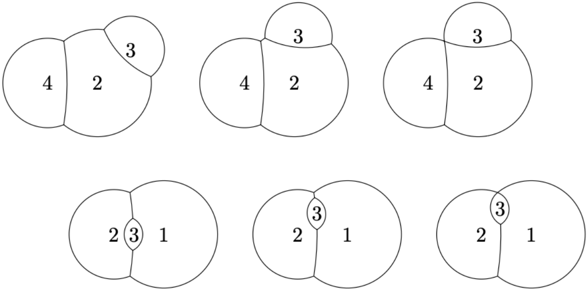
Double and Triple bubble on $\mathbb{R}^n/\mathbb{S}^n$





This already concludes proof of double/triple-bubble on $\mathbb{R}^n/\mathbb{S}^n$!
Quadruple-bubble needs more work...

For $q \gg 1$, leads to questions on incidence structure of $\{\Omega_i\}_{i=1,\ldots,q}$.
How to proceed? How did we conclude on $\mathbb{G}^n$ for all $q \leq n+1$?

The Isoperimetric Profile for Multi-Bubbles

$(M^n, g, \mu) \in \{\mathbb{G}^n, \mathbb{S}^n\}$. Need finite volume, so cannot work on $\mathbb{R}^n$.
$V(\Omega) = (V(\Omega_1), \ldots, V(\Omega_q)) \in \Delta^{(q-1)} := \left\{ v \in \mathbb{R}^q \,; v_i \geq 0 \,,\ \sum_{i=1}^q v_i = 1 \right\}$.

Isoperimetric Profile: $I^{(q-1)} : \Delta^{(q-1)} \to \mathbb{R}_+$,

$$I^{(q-1)}(v) := \inf \{A(\Omega); V(\Omega) = v\} .$$

Model Isoperimetric Profile: $I_m^{(q-1)} : \text{int}\, \Delta^{(q-1)} \to \mathbb{R}_+$,
(denoting by Ω^m the conjectured model standard q-cluster),

$$I_m^{(q-1)}(v) = A(\Omega^m) \text{ s.t. } V(\Omega^m) = v \in \text{int}\, \Delta^{(q-1)};$$

extend continuously to $\partial\Delta^{(q-1)}$.

Obviously $I^{(q-1)} \leq I_m^{(q-1)}$; want to show: $I^{(q-1)} \geq I_m^{(q-1)}$ on $\Delta^{(q-1)}$.
Inducting on q, can assume $I^{(q-1)} = I_m^{(q-1)}$ on the boundary $\partial\Delta^{(q-1)}$.

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Partial Differential Inequality for Profile

On $\mathbb{G}^n$, one can show that a fully non-linear elliptic PDE holds:

$$\mathrm{tr}((-\nabla^2 \mathcal{I}_m)^{-1}) = 2\mathcal{I}_m \text{ on } \Delta^{(q-1)}.$$

Similar (but more complicated) PDE holds on $\mathbb{S}^n$.

If we could show that the following PDI holds (in the viscosity sense):

$$\nabla^2 \mathcal{I} < 0 \ , \ \mathrm{tr}((-\nabla^2 \mathcal{I})^{-1}) \leq 2\mathcal{I} \text{ on } \mathrm{int}\,\Delta^{(q-1)},$$

since $\mathcal{I} = \mathcal{I}_m$ on $\partial\Delta^{(q-1)}$ by induction, $\mathcal{I} \geq \mathcal{I}_m$ by maximum-principle.

This is our global information!! PDI takes into account entire $\Delta^{(q-1)}$.
Key idea: instead of using global information in space parameters $\mathbb{G}^n$, PDI propagates global information in volume parameters $\Delta^{(q-1)}$

Hence, need upper bounds on $\nabla^2 \mathcal{I}(v)$ for a given $v \in \mathrm{int}\,\Delta^{(q-1)}$.
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The index-form $Q(X)$

Recall $\frac{d}{dt} F_t = X \circ F_t$ diffeo, $\Omega_t = F_t(\Omega)$, $\mathcal{I}(V(\Omega_t)) \leq A(\Omega_t)$. Hence:

$$\left\langle \nabla \mathcal{I}, \delta^1_X V \right\rangle = \delta^1_X A = \left\langle \lambda, \delta^1_X V \right\rangle \Rightarrow \nabla \mathcal{I} = \lambda.$$

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This generalizes stability: $\delta^1_X V = 0 \;\Rightarrow\; 0 \leq Q(X)$.

The goal: choose X well to get a sharp PDI for $\mathcal{I}$.

$Q(X)$ index-form, depends only on $f_{ij} = \left\langle X, \mathfrak{n}_{ij} \right\rangle$ on $\Sigma^1 = \sqcup_{i<j} \Sigma_{ij}$.

$$Q(f) = -\langle L_{Jac} f, f \rangle_{\Sigma^1} + \int_{\partial^* \Sigma^1} \mathrm{bdry}(f, \mathrm{II}).$$

L_{Jac} is the Jacobi operator:

$$-\delta^1_{f\mathfrak{n}} H_{\Sigma,\mu} = L_{Jac} f = \Delta_{\Sigma,\mu} f + (\mathrm{Ric}_{g,\mu}(\mathfrak{n},\mathfrak{n}) + \|\mathrm{II}\|^2) f.$$

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Obtaining the Partial Differential Inequality

$$(\delta_X^1 V)^T \, \nabla^2 \mathcal{I} \, \delta_X^1 V \leq -\langle L_{Jac} X^{\mathfrak{n}}, X^{\mathfrak{n}} \rangle_{\Sigma^1} + \int_{\partial^* \Sigma^1} \text{bdry}(X^{\mathfrak{n}}, \text{II}),$$

$$L_{Jac} X^{\mathfrak{n}} = \left(\Delta_{\Sigma,\mu} + \text{Ric}_{g,\mu}(\mathfrak{n},\mathfrak{n}) + \|\text{II}\|^2 \right) X^{\mathfrak{n}}.$$

Here $\text{Ric}_{g,\mu}(\mathfrak{n},\mathfrak{n}) = 1$ on $\mathbb{G}^n$ and $\text{Ric}_{g,\mu}(\mathfrak{n},\mathfrak{n}) = (n-1)$ on $\mathbb{S}^n$.
And we already know that $\text{II} = 0$ on $\mathbb{G}^n$ and $\text{II} = \kappa_{ij}\text{Id}$ on $\mathbb{S}^n$.

On $\mathbb{G}^n$: $L_{Jac} 1 = 1$, so if $X^{\mathfrak{n}_{ij}} = a_i - a_j$ then $L_{Jac} X^{\mathfrak{n}_{ij}} = a_i - a_j$.
As $\mathfrak{n}_{ij} + \mathfrak{n}_{jk} + \mathfrak{n}_{ki} = 0$ on $\partial^* \Sigma^1$, possible to (approximately) construct X.
This yields sharp PDI, and we conclude the proof that $\mathcal{I} \geq \mathcal{I}_m$.

On $\mathbb{S}^n$: fields yielding sharp PDI exist (non-trivial). But we don't have explicit formula, unless cluster is (pseudo)-conformally-flat ($\{\mathfrak{c}_i, \kappa_i\}$).
E.g.: • when cluster is full-dimenional, i.e. $\text{affine-rank}\{\mathfrak{c}_i\}_{i=1}^q = q - 1$;
• if all bubbles have a mutual common point.
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But what if the cluster is not pseudo-conformally-flat???
While this should never happen, we cannot a-priori exclude this.
Using Step 5 (= some tricks), we can go up to $q \leq 6$ on $\mathbb{S}^n$.

Obtaining the Partial Differential Inequality

$$(\delta_X^1 V)^T \, \nabla^2 \mathcal{I} \, \delta_X^1 V \leq -\langle L_{Jac} X^{\mathfrak{n}}, X^{\mathfrak{n}} \rangle_{\Sigma^1} + \int_{\partial^* \Sigma^1} \mathrm{bdry}(X^{\mathfrak{n}}, \mathrm{II}),$$

$$L_{Jac} X^{\mathfrak{n}} = \left(\Delta_{\Sigma,\mu} + \mathrm{Ric}_{g,\mu}(\mathfrak{n},\mathfrak{n}) + \|\mathrm{II}\|^2 \right) X^{\mathfrak{n}}.$$

Here $\mathrm{Ric}_{g,\mu}(\mathfrak{n},\mathfrak{n}) = 1$ on $\mathbb{G}^n$ and $\mathrm{Ric}_{g,\mu}(\mathfrak{n},\mathfrak{n}) = (n-1)$ on $\mathbb{S}^n$.
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Obtaining the Partial Differential Inequality

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Thank you for your attention!

Equal Volume Case in $\mathbb{S}^n$ and $\mathbb{R}^n$

Equal Volume Multi-Bubble on $\mathbb{S}^n$ (M.–Neeman '18)

On $\mathbb{S}^n$, for any $q \leq n+2$, if $V(\Omega_1) = \ldots = V(\Omega_q) = \frac{1}{q}$ then the unique minimizer is a standard bubble.

Proof: immediate consequence from $\mathbb{G}^n$, since spherical and Gaussian volume/area coincide for centered cones on $\mathbb{S}^n \subset \mathbb{G}^{n+1}$, and the unique equal volumes minimizer on $\mathbb{G}^{n+1}$ for $q \leq (n+1)+1$ is the centered simplicial cluster (whose cells are centered cones).

Equal Volume Triple-Bubble on $\mathbb{R}^3$ (Lawlor '22)

On $\mathbb{R}^3$, if $V(\Omega_1) = V(\Omega_2) = V(\Omega_3)$, then the unique (?) minimizer is a standard triple-bubble.

Jump back....