Optimal transport and quantitative geometric inequalities

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Isoperimetric Problems
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Plan of the talk

GOAL: discuss some recent geometric inequalities in their quantitative form for smooth Riemannian manifolds with OT tools, more precisely using techniques coming from non-smooth synthetic Ricci curvature lower bounds

- Quantitative Levy-Gromov isoperimetric inequality. (with F. Cavalletti and F. Maggi)
- Quantitative Obata's rigidity Theorem. (with F. Cavalletti and D. Semola)

Isoperimetric problem

One of oldest problems in mathematics, roots in myths of 2000 years ago (Queen Dido's problem).
Q: Given a space X and a volume v, what is the minimal amount of (boundary) area needed to enclose the volume $v > 0$?

Examples

- $X = \mathbb{R}^n \rightsquigarrow$ Euclidean isoperimetric inequality:
  For all $E \subset \mathbb{R}^n$ it holds $|\partial E| \geq |\partial B|$ where B is a round ball s.t. $|B| = |E|$.
- $X = \mathbb{S}^n$ analogous:
  For all $E \subset \mathbb{S}^n$ it holds $|\partial E| \geq |\partial B|$ where B is a metric ball (i.e. a spherical cap) s.t. $|B| = |E|$

RK: In both of the examples the space is fixed (Euclidean space of Sphere), such a space contains a model subset

Levy-Gromov inequality

Besides the euclidean one, probably the most famous isoperimetric inequality is the Levy-Gromov isoperimetric inequality:

Levy-Gromov Isoperimetric inequality

Let (M^n, g) be a Riemannian manifold with $Ric_g \geq (n-1)g$ and $E \subset M$ domain with smooth boundary ∂E.
Let $\mathbb{S}^n$ be the round sphere of unit radius (in particular $Ric \equiv n-1$), and $B \subset \mathbb{S}^n$ be a metric ball s.t. $\frac{|E|}{|M|} = \frac{|B|}{|\mathbb{S}^n|}$.
Then

$$\frac{|\partial E|}{|M|} \geq \frac{|\partial B|}{|\mathbb{S}^n|}$$

RK. (1) In the (LGI) the space is NOT fixed: any subset in any manifold with $Ric \geq n-1$ is compared with the model subset (i.e. spherical cap) in the model space (i.e. the sphere).
(2) (LGI) is global in the space, i.e. it does not depend just on

Equivalent way to state LG inequality in terms of isoperimetric profile

- Given a Riemannian manifold (M, g), define its *isoperimetric profile function* as

$$\mathcal{I}_{(M,g)}(v) := \inf\left\{\frac{|\partial E|}{|M|} \,:\, \frac{|E|}{|M|} = v\right\},\ \forall v \in [0,1].$$

- Levy-Gromov Inequality can be stated as: Given (M^n, g) with $Ric_g \geq (n-1)g$ then

$$\mathcal{I}_{(M,g)}(v) \geq \mathcal{I}_{\mathbb{S}^n}(v),\ \forall v \in [0,1].$$

Rigidity and almost rigidity in the Levy-Gromov inequality

- Rigidity: If there exists $E \subset M$ with $\frac{|E|}{|M|} = v \in (0,1)$ satisfying $\frac{|\partial E|}{|M|} = \mathcal{I}_{(M,g)}(v) = \mathcal{I}_{\mathbb{S}^n}(v)$, then
  1) $(M^n, g) \simeq \mathbb{S}^n$ isometric
  2) $E \simeq B$ metric ball.
- Question: Stability? i.e. If "$=$" in (LGI) is almost attained,
  Q1) What can we say on (M^n, g)? Is it close to a sphere? In which sense?
  Q2) What can we say on E? Is it close to a metric ball? In which sense?

About Question 1

THM 1 (Particular case of Berard-Besson-Gallot, Inv. Math. 1985) Given (M^n, g) with $Ric_g \geq (n-1)g$ and $\mathrm{diam}(M) = D$ (recall from Bonnet-Myers $D \in (0, \pi]$) then

$$\frac{\mathcal{I}_{(M,g)}(v)}{\mathcal{I}_{\mathbb{S}^n}(v)} \geq \left(\frac{\int_0^{\pi/2} (\cos t)^{n-1} dt}{\int_0^{D/2} (\cos t)^{n-1} dt} \right)^{1/n}, \quad \forall v \in (0,1)$$

RK: 1) E. Milman improved THM 1 to a sharp version (JEMS '15).
2) rhs is ≥ 1 so the result sharpens the classical LGI
3) It follows that there exists $C_{n,v} > 0$ such that if for some $v \in (0,1)$ it holds $\mathcal{I}_{(M,g)}(v) \leq \mathcal{I}_{\mathbb{S}^n}(v) + \delta$, then

$$\pi - D \leq C_{n,v} \delta^{1/n}.$$

$\rightsquigarrow$ $d_{GH}(M, S(X)) \leq \varepsilon(\delta)$ by Cheeger-Colding Almost Maximal Diameter Thm (Annals of Math. 1996)

Answering Question 2 in Euclidean setting

Quantitative Euclidean Isoperimetric Inequality
(Fusco-Maggi-Pratelli, Annals of Math. 2008)
There exists $C_n > 0$ such that for every $E \subset \mathbb{R}^n$ there exists a round ball $B \subset \mathbb{R}^n$ with $|E| = |B|$ and

$$\frac{|E \Delta B|}{|E|} \leq C_n \left(\frac{|\partial E|}{|\partial B|} - 1 \right)^{1/2}$$

RK: 1) Of course it implies EII. The rhs is the so-called "isoperimetric deficit" and is zero iff E is a ball.
2) The proof of FMP is via a "quantitative symmetrization".
3) Alternative proof of the result via Brenier L^2-Optimal Transport map (by Figalli-Maggi-Pratelli, Inv. Math. 2010) and via regularity theory and selection principle (Cicalese-Leonardi, ARMA 2012).

Answering Question 2 in spherical setting

Quantitative Spherical Isoperimetric Inequality
(Bogelein-Duzaar-Fusco, Adv. Calc. Var. 2015)
For every $v \in (0,1)$ and every $n \geq 2$ there exists $C_{n,v} > 0$ with the following property.
For every $E \subset \mathbb{S}^n$ with $\frac{|E|}{|\mathbb{S}^n|} = v$ there exists a metric ball $B \subset \mathbb{S}^n$ with $|B| = |E|$ such that

$$|E \Delta B| \leq C_{n,v} \left(\frac{|\partial E|}{|\mathbb{S}^n|} - \mathcal{I}_{\mathbb{S}^n}(v) \right)^{1/2}$$

Proof: along the same lines of Cicalese-Leonardi's selection principle.

Difficulties about Question 2: quantitative Levy-Gromov inequality

The above quantitative isoperimetric inequalities are for a fixed space ($\mathbb{R}^n$ or $\mathbb{S}^n$), with the highest possible degree of symmetry.

LGI is for any (M^n, g) with $Ric_g \geq (n-1)g$

$\rightsquigarrow$ No fixed space and no symmetry.

$\rightsquigarrow$ The above approaches seem not to be applicable:

- Symmetrization (FMP): since M is not symmetric it makes little sense to speak of symmetrization in M.
- Brenier Map, L^2-OT (FMP): works in $\mathbb{R}^n$ but already in $\mathbb{S}^n$ it is an open problem to prove Spherical Isoperimetric Inequality via Brenier Map.
- Selection Principle (CL): would need smooth convergence of metrics while here the natural convergence is Gromov-Hausdorff.

Brief history of localization

The localization technique is a way to reduce an a-priori complicated high dimensional problem to a family of simpler 1-dimensional problems.

- In $\mathbb{R}^n$ or $\mathbb{S}^n$, using the high symmetry of the space, 1-D localizations can be usually obtained via iterative bisections
  - Roots in a paper by Payne-Weinberger '60 about sharp estimate of 1^{st} eigenvalue of Neumann Laplacian in compact convex sets of $\mathbb{R}^n$
  - Formalized by Gromov-V. Milman '87, Kannan - Lovász - Simonovits '95
- Extended by B. Klartag '14 to Riemannian manifolds via L^1-optimal trasport: no symmetry but still heavily using the smoothness of the space (estimates on 2^{nd} fundamental form of level sets of the Kantorovich potential φ)

The result: quantitative Levy-Gromov inequality

THM 2 (Cavalletti-Maggi-M., CPAM 2019)
For every $v \in (0,1)$ and $n \geq 2$ there exists $C_{n,v} > 0$ with the following properties.
Let (M^n, g) be with $Ric_g \geq (n-1)g$. For every $E \subset M$ with $\frac{|E|}{|M|} = v$ there exists a metric ball $B \subset M$ with $|B| = |E|$ such that

$$|E \Delta B| \leq C_{n,v} \left(\frac{|\partial E|}{|M|} - \mathcal{I}_{\mathbb{S}^n}(v) \right)^{\frac{n}{n^2+n-1}}$$

In particular, if $E \subset M$ is an isoperimetric subset with $\frac{|E|}{|M|} = v$, then

$$|E \Delta B| \leq C_{n,v} \left(\mathcal{I}_{(M,g)}(v) - \mathcal{I}_{\mathbb{S}^n}(v) \right)^{\frac{n}{n^2+n-1}}$$

RK Difference with (QEII) or (QSII): here $E \subset M$ and $|\partial E|$ is compared with $\mathcal{I}_{\mathbb{S}^n}$ (not with $\mathcal{I}_{(M,g)}$) via a "Levy-Gromov isoperimetric deficit".

The result holds in higher generality

Actually we prove THM1 and THM 2 more generally for essentially non-branching $CD(N-1, N)$ metric measure spaces. These are (a priori) non-smooth spaces of dimension $\leq N$ and Ricci $\geq N-1$ in a synthetic sense via OT (Lott-Sturm-Villani).

Examples entering this class of spaces:

- Weighted manifolds with N-Bakry-Émery Ricci tensor bounded below by $N-1$
- Measured Gromov Hausdorff limits of Riemannian N-dimensional manifolds satisfying $Ric_g \geq (N-1)g$ and more generally the class of $RCD(N-1, N)$ spaces.
- Finite dimensional Alexandrov spaces with curvature ≥ 1
- Finsler manifolds satisfying $CD(N-1, N)$

PART 2:
A QUANTITATIVE OBATA THEOREM

Spectral gap

Sharp Lichnerowicz spectral gap: Let (M, g) be n-dim with Ric$\geq n-1$ and let $f \in Lip(M)$ with $\int_M f \, dvol_g = 0$ then

$$\int_M f^2 \, dvol_g \leq \frac{1}{n} \int_M |\nabla f|^2 \, dvol_g .$$

- Given (M, g), the first non-zero eigenvalue of the Neumann Laplacian is:

$$\lambda_1(M) := \inf \left\{ \int_M |\nabla f|^2 dvol_g \; : \; \|f\|_{L^2(M)} = 1, \int_M f \, dvol_g = 0 \right\}$$

- Lichnerowicz inequality can be stated as: let (M, g) be n-dim with Ric$\geq n-1$, then

$$\lambda_1(M) \geq n = \lambda_1(\mathbb{S}^n)$$

Rigidity and Stability of Lichnerowicz inequality

- Rigidity: Obata's Theorem 1962
  Let (M, g) be n-dim with Ric$\geq n-1$.
  Then $\lambda_1(M) = n$ iff (M, g) is isometric to $\mathbb{S}^n$.
  Note: First eigenfunction on $\mathbb{S}^n$ is $\sqrt{n+1}\cos(\mathrm{d}_x)$, $\forall x \in \mathbb{S}^n$
- Stability? i.e. if "$=$" in spectral gap is almost attained:
  - Cheng '75, Croke '82: $\lambda_1(M) \simeq n$ iff $diam(M) \simeq \pi$
  - Berard-Besson-Gallot '85:
    $\lambda_1(M) - n \geq C_n(\pi - diam(M))^n$
  - Bertrand '07: stability of eigenfunctions: there exists a function $\tau(t) \to 0$ as $t \to 0$ s.t.
    if $\lambda_1(M) \leq n + \epsilon$, then $\|f - \sqrt{n+1}\cos(\mathrm{d}_x)\|_\infty \leq \tau(\epsilon)$ for f first eigenfunction.
- Question: can we make quantitative Bertrand's result and generalize it to a function with almost optimal Rayleigh quotient (but non-necessarily eigenfunction)?
  i.e. if

The result: Quantitative Obata's Theorem

THM(Cavalletti-M.-Semola, Analysis & PDE 2022)
For every $n \geq 2$ there exists $C_n > 0$ with the following properties.
Let (M, g) be n-dim with Ric$\geq n-1$. For every $f \in Lip(M)$ with

$$\int_M f \, dvol_g = 0, \quad \int_M f^2 \, dvol_g = 1,$$

there exists a point $x \in M$ such that

$$\|f - \sqrt{n+1}\cos(\mathrm{d}_x)\|_2 \leq C_n \left(\int_M |\nabla f|^2 \, dvol_g - n \right)^{\frac{1}{6n+4}}.$$

In particular, if f is a first eigenfunction, then

$$\|f - \sqrt{n+1}\cos(\mathrm{d}_x)\|_2 \leq C_n \left(\lambda_1(M) - \lambda_1(\mathbb{S}^n)\right)^{\frac{1}{6n+4}}.$$

RK: Proved more generally for essentially non branching $CD(N-1, N)$ spaces.

PART 3:
SOME IDEAS OF THE PROOFS

Technique: 1-D localization

Let $(X, \mathsf{d}, \mathfrak{m})$ be e.n.b. $CD(K,N)$, with $\mathfrak{m}(X) = 1$. Given $E \subset X$ we can find a "1-D localization" $\{X_\alpha\}_{\alpha\in Q}$ of X, i.e.

1. $\{X_\alpha\}_{\alpha\in Q}$ is (essentially) a partition of X, i.e. $\mathfrak{m}(X \setminus \mathring{\bigcup}_{\alpha\in Q} X_\alpha) = 0$
2. Disintegration of $\mathfrak{m}$ wrt $\{X_\alpha\}_{\alpha\in Q}$ (kind of non-straight Fubini Thm): $\mathfrak{m} = \int_Q \mathfrak{m}_\alpha \, \mathfrak{q}(d\alpha)$, with $\mathfrak{q}(Q) = 1$ and $\mathfrak{m}_\alpha(X_\alpha) = \mathfrak{m}_\alpha(X) = 1$ for $\mathfrak{q}$-a.e. $\alpha \in Q$
3. X_α is a geodesic in X and $(X_\alpha, |\cdot|, \mathfrak{m}_\alpha)$ is a $CD(K,N)$ space
4. $\mathfrak{m}_\alpha(E \cap X_\alpha) = \mathfrak{m}(E)$, for $\mathfrak{q}$-a.e. $\alpha \in Q$

How to obtain a localization: Consider the OT-problem with $c(x,y) = \mathsf{d}(x,y)$ between

$$\mu_0 := (\chi_E/\mathfrak{m}(E))\mathfrak{m} \quad \text{and} \quad \mu_1 := (\chi_{X\setminus E}/\mathfrak{m}(X\setminus E))\mathfrak{m}.$$

X_α will be integral curve of $-\nabla\varphi$, with φ Kantorovich

More on how to construct a 1-D localization

- Recall that $\mathfrak{m}(X) = 1$, fix $E \subset X$ with $\mathfrak{m}(E) \in (0,1)$,
- Let $\mu_0 := \frac{\chi_E}{\mathfrak{m}(E)}\,\mathfrak{m}$ and $\mu_1 := \frac{1-\chi_E}{1-\mathfrak{m}(E)}\,\mathfrak{m} = \frac{\chi_{X\setminus E}}{\mathfrak{m}(X\setminus E)}\,\mathfrak{m}$
- Consider the L^1-optimal transport problem

$$\inf_{\gamma} \left\{ \int_{X\times X} \mathsf{d}(x,y)\, d\gamma \;:\; \gamma \in \mathcal{P}(X\times X), (\pi_1)_\sharp\gamma = \mu_0, (\pi_2)_\sharp\gamma = \mu$$

- By Optimal Transport techniques there exists a minimizer $\gamma \in \mathcal{P}(X\times X)$ and a 1-Lipschitz function $\varphi : X \to \mathbb{R}$ called Kantorovich potential such that, denoted

$$\Gamma := \{(x,y) \in X\times X \;:\; \varphi(x) - \varphi(y) = \mathsf{d}(x,y)\},$$

  γ is concentrated on Γ.
- The relation $\sim$ on X given by $x \sim y$ iff $(x,y) \in \Gamma$ or $(y,x) \in \Gamma$ is an equivalence relation on X (up to an $\mathfrak{m}$-negligible subset) and the equivalence classes are geodesics. $\rightsquigarrow$ partition of X into geodesics driven by E

Why L^1-trasport?

- It is more standard to consider the L^2-optimal transport problem: given $\mu_0, \mu_1 \in \mathcal{P}(X)$ let
$$\inf_{\gamma} \left\{ \int_{X \times X} \mathsf{d}(x,y)^2 \, d\gamma \ : \ \gamma \in \mathcal{P}(X \times X), (\pi_1)_\sharp \gamma = \mu_0, (\pi_2)_\sharp \gamma =$$
Which defines a metric W_2 on $\mathcal{P}(X)$.
- If $(\mu_t)_{t \in [0,1]}$ is a W_2-geod from μ_0 to μ_1, then μ_t concentrates on t-intermediate points of geodesics from *supp*(μ_0) to *supp*(μ_1): $\mu_t(\{\gamma(t) \ : \gamma \, \textit{geod}, \gamma(0) \in \mathrm{supp}(\mu_0), \gamma(1) \in \mathrm{supp}(\mu_1)\}) = 1$,
- moreover, from d^2-monotonicity, if γ_1 and γ_2 are such geodesics with $\gamma_1(0) \neq \gamma_2(0)$ then $\gamma_1(t) \neq \gamma_2(t)$ in a.e. sense. $\rightsquigarrow$ the L^2-transport at time t is given by an ess. inj. map.
- BUT it may happen $\gamma_1(s) = \gamma_2(t)$ for $s \neq t$
$\rightsquigarrow$ L^2-transport does not induce an equivalence relation

Levy-Gromov inequality via Localization

Let $(X, \mathsf{d}, \mathfrak{m})$ be an e.n.b. $CD(N-1, N)$ space.
Assume that for $E \subset X$ we can find a 1-D localization as above then

$$
\begin{aligned}
\mathfrak{m}^+(E) &:= \liminf_{\varepsilon\to 0^+} \frac{\mathfrak{m}(E^\varepsilon)-\mathfrak{m}(E)}{\varepsilon} \\
&= \liminf_{\varepsilon\to 0^+} \int_Q \frac{\mathfrak{m}_\alpha(E^\varepsilon)-\mathfrak{m}_\alpha(E)}{\varepsilon}\,\mathfrak{q}(d\alpha) \quad \text{by 2.} \\
&\geq \int_Q \liminf_{\varepsilon\to 0^+} \frac{\mathfrak{m}_\alpha(E^\varepsilon\cap X_\alpha)-\mathfrak{m}_\alpha(E\cap X_\alpha)}{\varepsilon}\,\mathfrak{q}(d\alpha) \quad \text{by 2.} \\
&\geq \int_Q \liminf_{\varepsilon\to 0^+} \frac{\mathfrak{m}_\alpha((E\cap X_\alpha)^\varepsilon\cap X_\alpha)-\mathfrak{m}_\alpha(E\cap X_\alpha)}{\varepsilon}\,\mathfrak{q}(d\alpha), \\
&\qquad \text{by } E^\varepsilon\cap X_\alpha \supset (E\cap X_\alpha)^\varepsilon\cap X_\alpha \\
&\geq \int_Q \mathfrak{m}_\alpha^+(E\cap X_\alpha)\,\mathfrak{q}(d\alpha) \\
&\geq \int_Q \mathcal{I}_{\mathbb{S}^N}(\mathfrak{m}_\alpha(E\cap X_\alpha))\,\mathfrak{q}(d\alpha) \text{ by 3.+Smooth LGI in 1D} \\
&= \int_Q \mathcal{I}_{\mathbb{S}^N}(\mathfrak{m}(E))\,\mathfrak{q}(d\alpha) \text{ by 4.} = \mathcal{I}_{\mathbb{S}^N}(\mathfrak{m}(E)).
\end{aligned}
$$

Quantitative Levy-Gromov: one dimensional estimates

- Let (M^n, g) be with Ric$\geq n-1$ and let $\mathfrak{m} = vol_g/|M|$. Given $E \subset M$ with $\mathfrak{m}(E) = v \in (0,1)$, we have:

$$0 \leq \delta \quad := \mathfrak{m}^+(E) - \mathcal{I}_{\mathbb{S}^n}(v) \quad \text{"Levy-Gromov isoperimetric deficit}$$
$$\geq \int_Q (\mathfrak{m}^+_\alpha(E \cap X_\alpha) - \mathcal{I}_{\mathbb{S}^n}(v))\, \mathfrak{q}(d\alpha) = \int_Q \delta_\alpha \mathfrak{q}(d\alpha).$$

- Since $(X_\alpha, \mathsf{d}, \mathfrak{m}_\alpha)$ is $CD(n-1, n)$ and $\mathfrak{m}_\alpha(E \cap X_\alpha) = \mathfrak{m}(E) = v$ (by 4.) $\Rightarrow 0 \leq \delta_\alpha := \mathfrak{m}^+_\alpha(E \cap X_\alpha) - \mathcal{I}_{\mathbb{S}^n}(v) =$ "1-dim Isop. Deficit"
- The 1-dim deficit δ_α controls $\pi - |X_\alpha|$:

$$\int_Q (\pi - |X_\alpha|)^n \, \mathfrak{q}(d\alpha) \leq C(n,v)\delta.$$

- RK: so far, also in the proof of Levy Gromov, no role of OT: works for any 1-D localization.

Quantitative Levy-Gromov: set of long rays

- Fix the set of long rays

  $$Q_{long} := \{\alpha \in Q : (\pi - |X_\alpha|)^n \leq \sqrt{\delta}\} \simeq \{\alpha \in Q : \delta_\alpha \leq \sqrt{\delta}\},$$

  so that (from last slide) $\mathfrak{q}(Q_{long}) \geq 1 - C(n, v)\sqrt{\delta}$
- Problem: we know that most rays have length $\sim \pi$, but how do they combine together?
  Is there are a "common south/north pole"?
  NO for a general 1-D localization. However in our case
  Exploit the variational character of the localization via OT.

Quantitative Levy-Gromov: structure of transport set

- Recall that for the set of long rays $\mathfrak{q}(Q_{long}) \geq 1 - C(n,v)\sqrt{\delta}$:

  $$Q_{long} := \{\alpha \in Q : (\pi - |X_\alpha|)^n \leq \sqrt{\delta}\} \simeq \{\alpha \in Q : \delta_\alpha \leq \sqrt{\delta}\},$$

- From cyclical d-monotonicity of the transport set, we get

  $$\begin{aligned} 2\pi \quad - \quad & \mathsf{d}(a(X_\alpha), b(X_\alpha)) - \mathsf{d}(a(X_{\bar\alpha}), b(X_{\bar\alpha})) \\ & \geq 2\pi - \mathsf{d}(a(X_\alpha), b(X_{\bar\alpha})) - \mathsf{d}(a(X_{\bar\alpha}), b(X_\alpha)) \end{aligned}$$

  Rearranging, for $\alpha, \bar\alpha \in Q_{long}$ gives

  $$2\delta^{\frac{1}{2n}} \geq (\pi - \mathsf{d}(a(X_\alpha), b(X_{\bar\alpha}))) + (\pi - \mathsf{d}(a(X_{\bar\alpha}), b(X_\alpha)))$$

- Using Ric$\geq n-1$, setting $P_N := a(X_{\bar\alpha}), P_S := b(X_{\bar\alpha})$, we get

  $$\mathsf{d}(a(X_\alpha), P_N) + \mathsf{d}(b(X_\alpha), P_S) \leq C(n, v)\delta^{\frac{1}{\ }}, \ \forall \alpha \in Q$$

Quantitative Levy-Gromov: constructing the metric ball

- Using 1-dim (LGI), for $\alpha \in Q_{long}$, calling $E_\alpha := X_\alpha \cap E$ it holds
  $$\min\{\mathfrak{m}_\alpha(E_\alpha \Delta [0, r_v]), \mathfrak{m}_\alpha(E_\alpha \Delta [|X_\alpha| - r_v, |X_\alpha|])\} \leq \delta_\alpha \leq \sqrt{\delta}$$
  where r_v is s.t. $\mathfrak{m}_{\mathbb{S}^n}(B_{r_v}) = v$.
- So we can write $E = E_N \cup E_S \cup E_{err}$ with:
  $\mathfrak{m}(E_{err}) \leq C(n, v)\sqrt{\delta}$,
  $E_N := \{x \in E_\alpha \ E_\alpha \simeq [0, r_v]\}$,
  $E_S := \{x \in E_\alpha \ E_\alpha \simeq [|X_\alpha| - r_v, |X_\alpha|]$
- Using relative isoperimetric inequality inside $B_\varepsilon(P_N)$ (or in $B_\varepsilon(P_S)$) with $\varepsilon \ll r_v$, we get
  $$\min\{\mathfrak{m}(E_N), \mathfrak{m}(E_S)\} \leq C(n, v)\delta^{\frac{1}{n}}$$
- Putting all together:

Quantitative Obata's Theorem

- Given (M^n, g) with Ric$\geq n-1$, and $f : M \to \mathbb{R}$ with $\int_M f\, \mathfrak{m} = 0$, $\int_M f^2\, \mathfrak{m} = 1$, associate a 1D-localization:

$$\mathfrak{m} = \int_Q \mathfrak{m}_\alpha \mathfrak{q}(d\alpha), \ \int f\, \mathfrak{m}_\alpha = 0, \ (X, \mathrm{d}, \mathfrak{m}_\alpha) \in CD(n{-}1, n)$$

- Recalling that $\mathfrak{m}(M) = 1$, $\lambda_1(\mathbb{S}^n) = n$, let

$$\begin{aligned} 0 \leq \delta \ &:= \int_M (|\nabla f|^2 - n)\mathfrak{m} = \text{“Spectral deficit”} \\ &\geq \int_Q \left(\int_{X_\alpha} ((f|_{X_\alpha})')^2 - n)\mathfrak{m}_\alpha \right) \mathfrak{q}(d\alpha) = \int_Q \delta_\alpha c_\alpha^2 \mathfrak{q}(d\alpha \end{aligned}$$

where $c_\alpha = \|f\|_{L^2(\mathfrak{m}_\alpha)}$.

- New difficulties:
  1) show that $c_\alpha \geq c > 0$ for “most” α, up to $\mathfrak{q}$-meas$\leq \delta$
  2) show that $c_\alpha \simeq c_{\bar{\alpha}}$ for “most” α, up to $\mathfrak{q}$-meas$\leq \delta$.

!!THANK YOU FOR THE ATTENTION!!