Capillary Surfaces and a Model of Nanowire Growth

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(based on joint works with G. DE PHILIPPIS, I. FONSECA, N. FUSCO, G. LEONI)

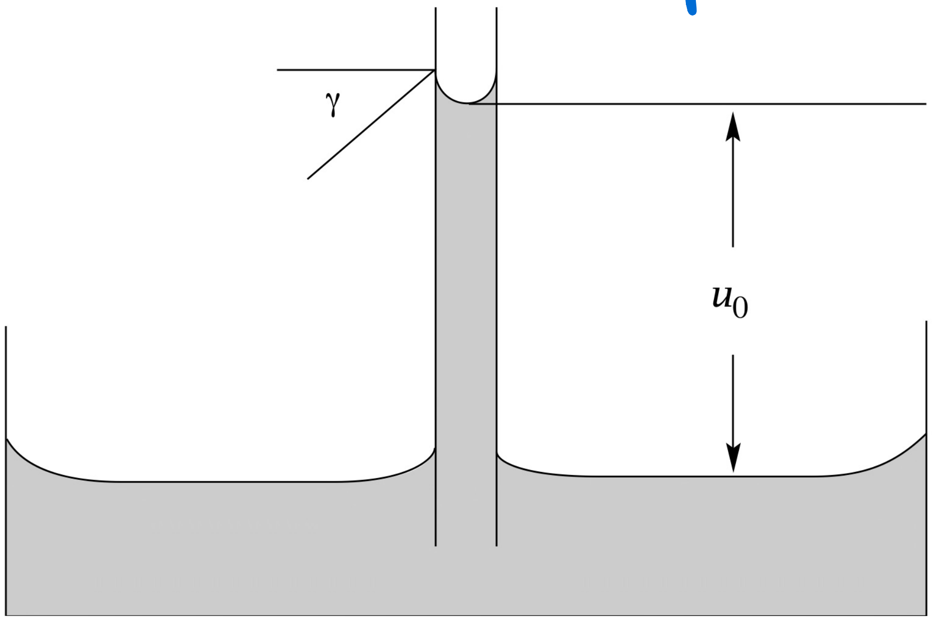
Capillarity Phenomena

From FINN's BOOK:

Capillarity phenomena are all about us; anyone who has seen a drop of dew on a plant leaf or the spray from a waterfall has observed them. Apart from their frequently remarked poetic qualities, phenomena of this sort are so familiar as to escape special notice. In this sense the rise of liquid in a narrow tube is a more dramatic event that demands and at first defied explanation; recorded observations of this and similar occurrences can be traced back to times of antiquity, and for lack of explanation came to be described by words deriving from the Latin word "capillus", meaning hair.

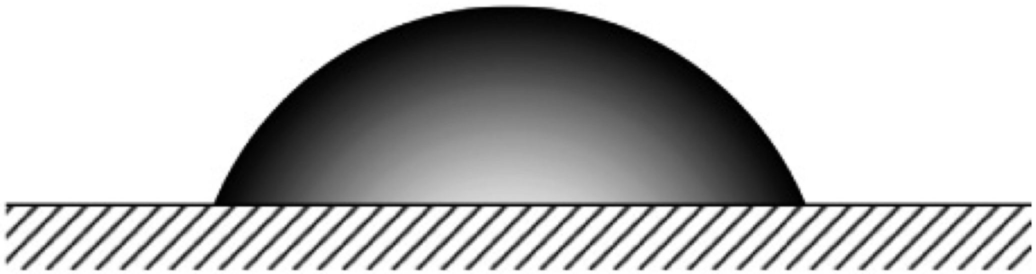
It was not until the eighteenth century that an awareness developed that these and many other phenomena are all manifestations of something that happens whenever two different materials are situated adjacent to each other and do not mix. If one (at least) of the materials is a fluid, which forms with another fluid (or gas) a free surface interface, then the interface will be referred to as a *capillary surface*.

Capillarity Phenomena

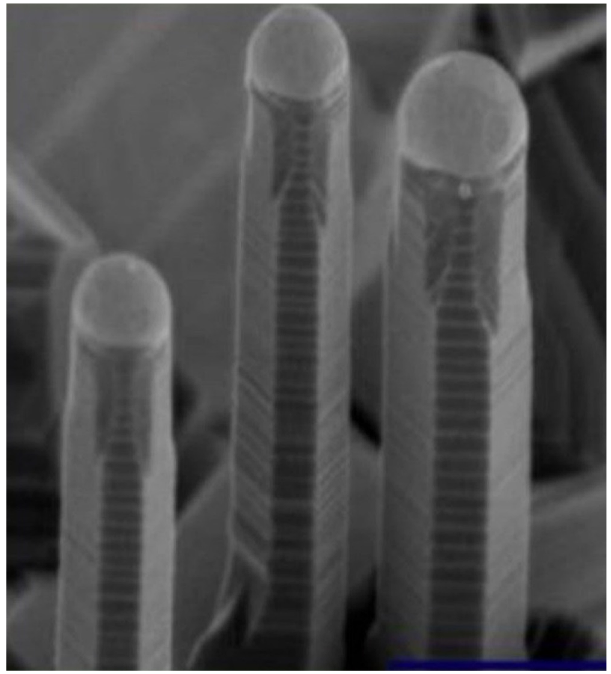


CAPILLARITY TUBE

PENDENT DROP



SESSILE DROP

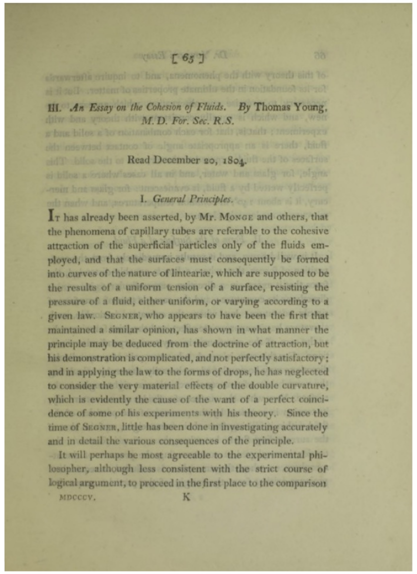


NANO TUBES

A bit of history

The study of CAPILLARITY PHENOMENA has a long tradition:

LEONARDO



[65]

III. *An Essay on the Cohesion of Fluids.* By Thomas Young, *M. D. For. Sec. R. S.*

Read December 20, 1804.

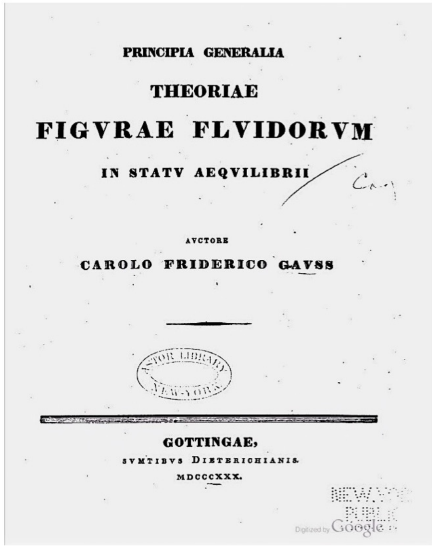
I. *General Principles.*

It has already been asserted, by Mr. Monge and others, that the phenomena of capillary tubes are referable to the cohesive attraction of the superficial particles only of the fluids employed, and that the surfaces must consequently be formed into curves of the nature of lintearíæ, which are supposed to be the results of a uniform tension of a surface, resisting the pressure of a fluid, either uniform, or varying according to a given law. Segner, who appears to have been the first that maintained a similar opinion, has shown in what manner the principle may be deduced from the doctrine of attraction, but his demonstration is complicated, and not perfectly satisfactory; and in applying the law to the forms of drops, he has neglected to consider the very material effects of the double curvature, which is evidently the cause of the want of a perfect coincidence of some of his experiments with his theory. Since the time of Segner, little has been done in investigating accurately and in detail the various consequences of the principle.

It will perhaps be most agreeable to the experimental philosopher, although less consistent with the strict course of logical argument, to proceed in the first place to the comparison

MDCCCV. K

YOUNG



PRINCIPIA GENERALIA

THEORIAE

FIGVRAE FLVIDORVM

IN STATV AEQVILIBRII

AVCTORE

CAROLO FRIDERICO GAVSS

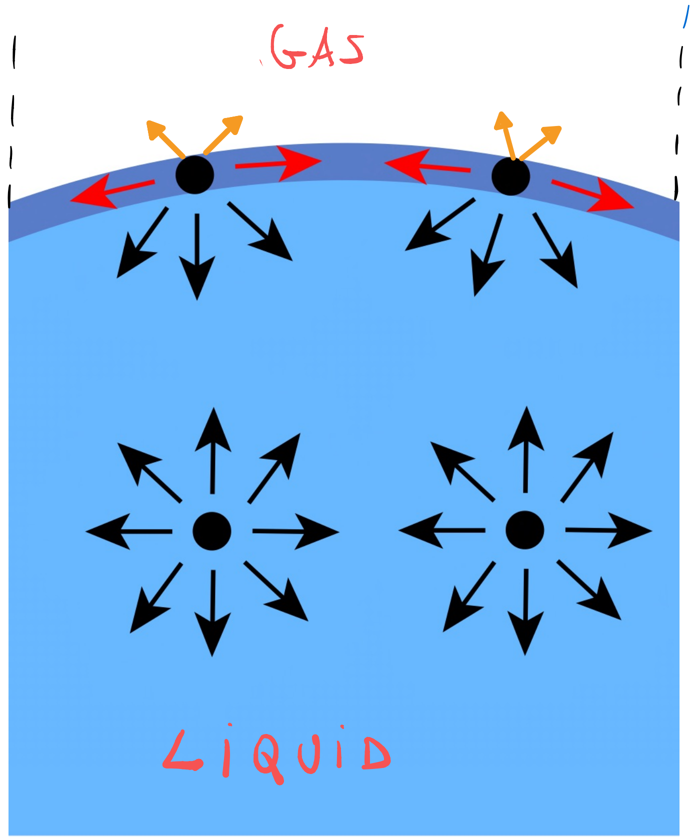
GOTTINGAE,

SVMTIBVS DIETERICHIANIS.

MDCCCXXX.

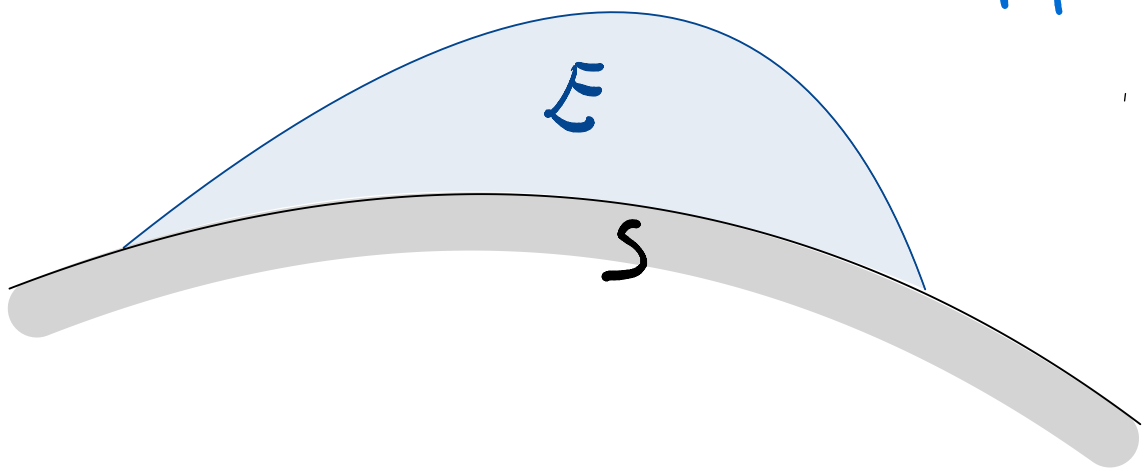
GAUSS

Gauss' variational approach

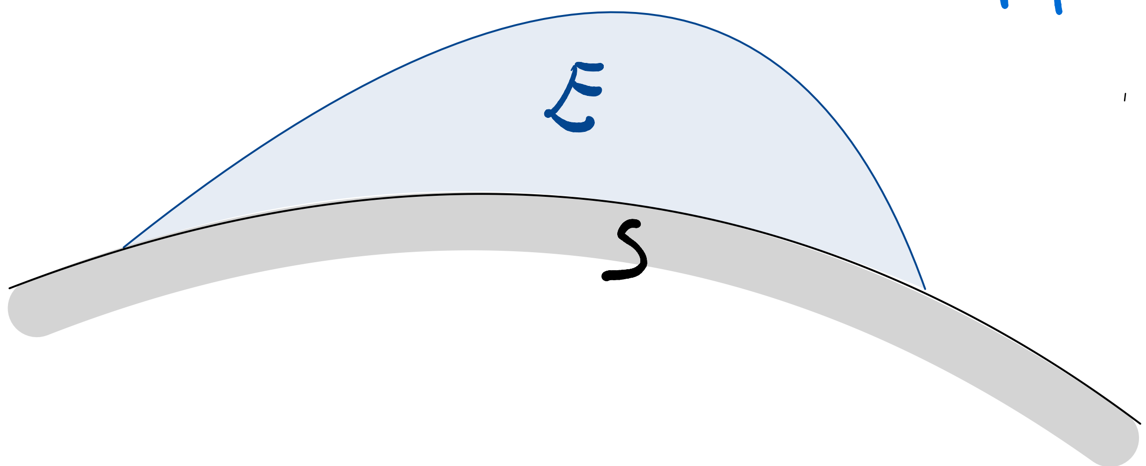


COHESIVE FORCES between liquid molecules are stronger than between liquid-air molecules ⟿ INTERNAL PRESSURE, MINIMIZATION OF INTERFACIAL AREA

Gauss' variational approach



Gauss' variational approach

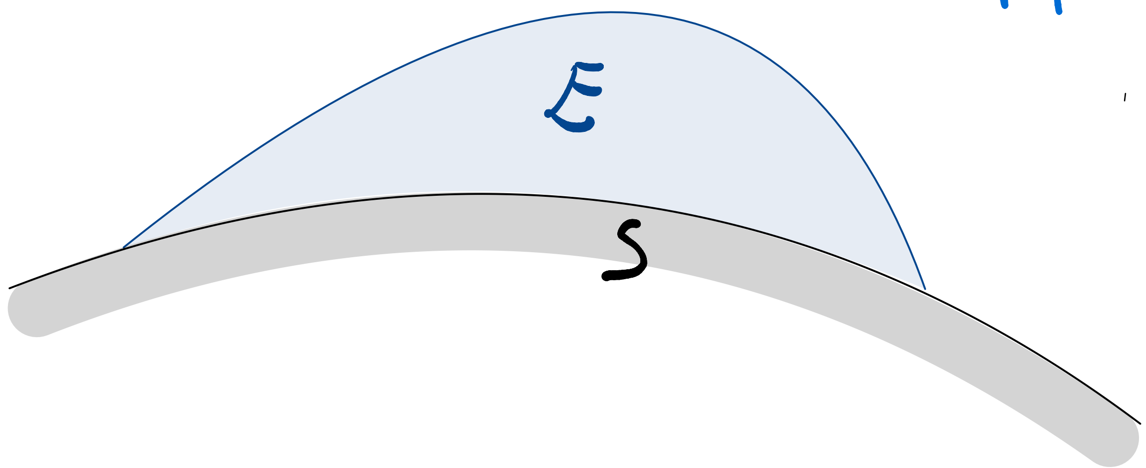


$$\boxed{\mathcal{E}(E) = \gamma_{LV}\,\sigma(\partial E \setminus S) + \gamma_{LS}\,\sigma(\partial E \cap \partial S) + \gamma_{SV}\,\sigma(\partial S \setminus \partial E)}$$

where $\sigma =$ HAUSDORFF MEASURE $\mathcal{H}^2$

$\gamma_{LV}, \gamma_{LS}, \gamma_{SV} =$ SURFACE TENSIONS

Gauss' variational approach

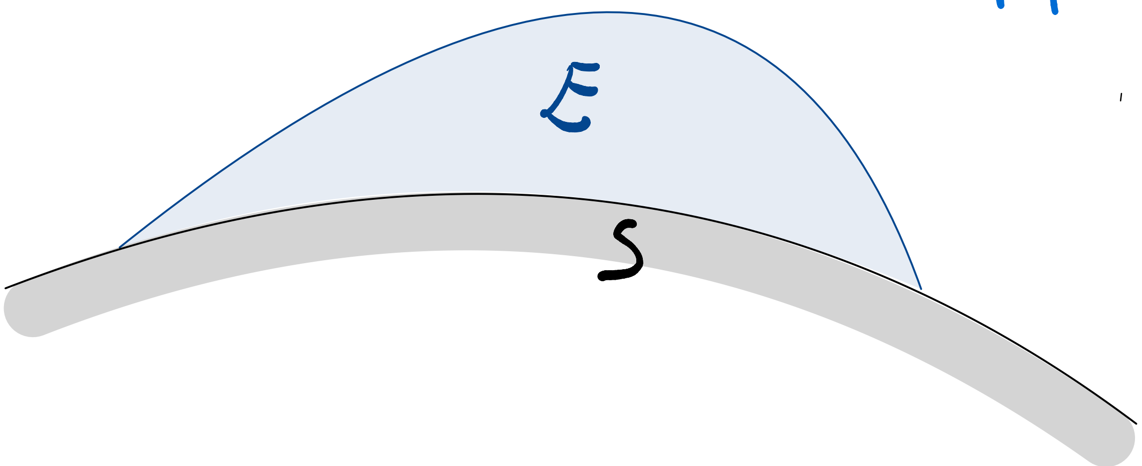


$$\boxed{\mathcal{E}(E) = \gamma_{LV}\,\sigma(\partial E \setminus S) + (\gamma_{LS} - \gamma_{SV})\,\sigma(\partial S \cap \partial E)} + \gamma_{SL}\,\sigma(\partial S)$$

where σ = HAUSDORFF MEASURE $\mathcal{H}^2$

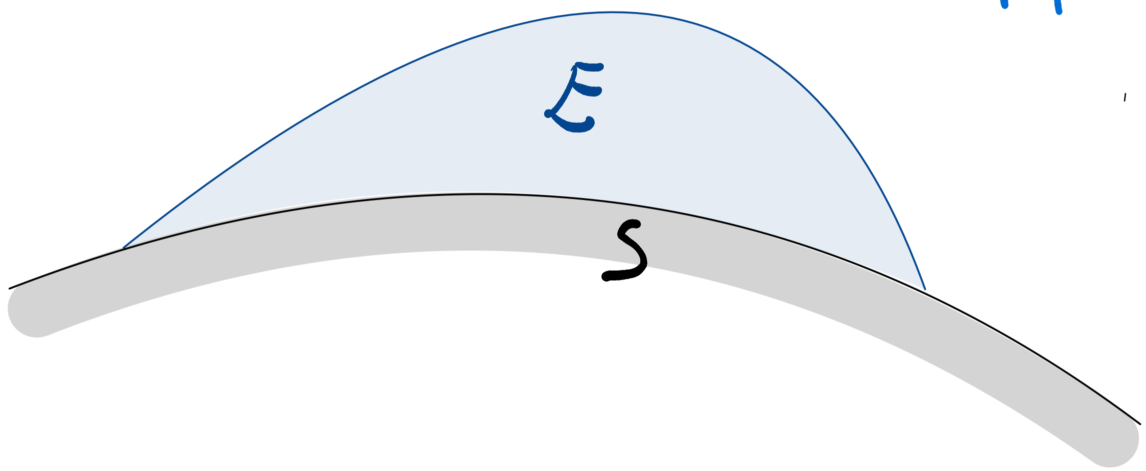
$\gamma_{LV}, \gamma_{LS}, \gamma_{SV}$: SURFACE TENSIONS

Gauss' variational approach



$$\mathcal{E}'(E) = \sigma(\partial E \setminus S) + \frac{\gamma_{LS} - \gamma_{SV}}{\gamma_{LV}}\,\sigma(\partial S \cap \partial E) + \frac{\gamma_{SL}}{\gamma_{LV}}\,\sigma(\partial S)$$

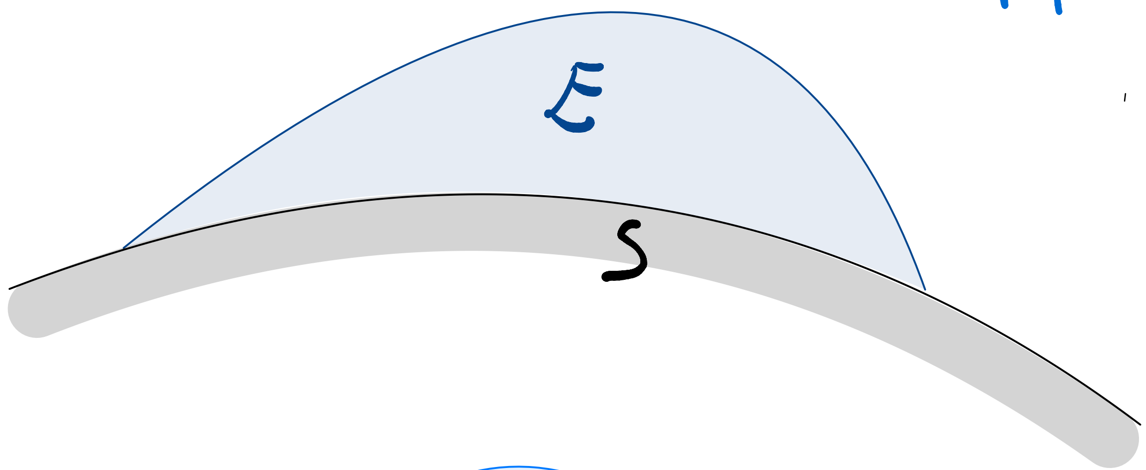
Gauss' variational approach



$$\boxed{\mathcal{E}'(E) = \sigma(\partial E \setminus S) + \frac{\gamma_{LS} - \gamma_{SV}}{\gamma_{LV}}\sigma(\partial S \cap \partial E)}$$

Assume the wetting condition $|\gamma_{LS} - \gamma_{SV}| < \gamma_{LV}$

Gauss' variational approach

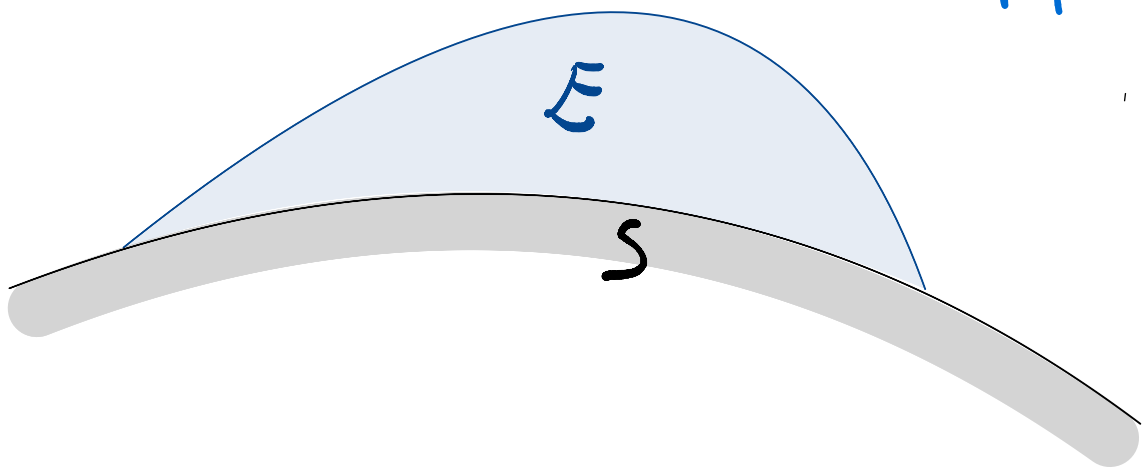


$$\mathcal{E}'(E) = \sigma(\partial E \setminus S) + \frac{\gamma_{LS} - \gamma_{SV}}{\gamma_{LV}} \sigma(\partial S \cap \partial E)$$

$\frac{\gamma_{LS} - \gamma_{SV}}{\gamma_{LV}} =: -\lambda$

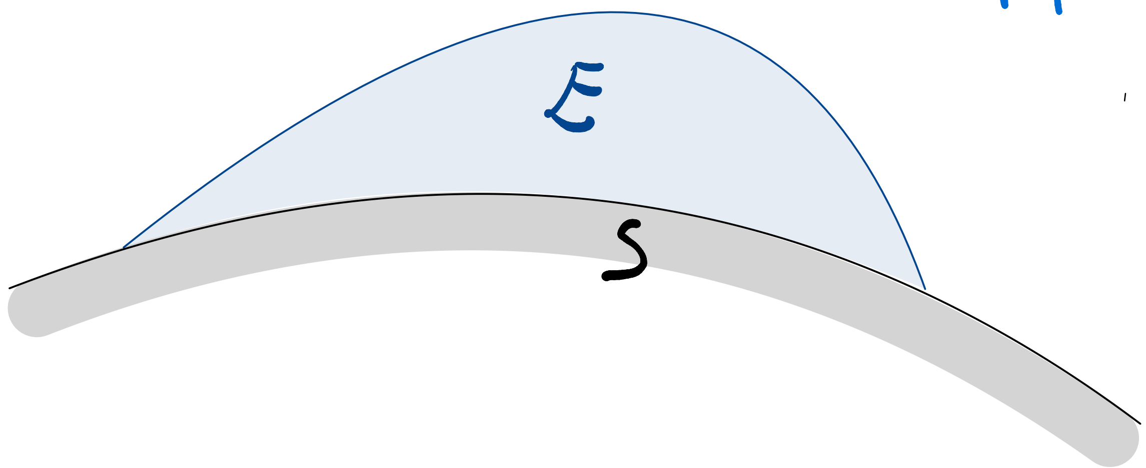
Assume the wetting condition $|\gamma_{LS} - \gamma_{SV}| < \gamma_{LV}$

Gauss' variational approach



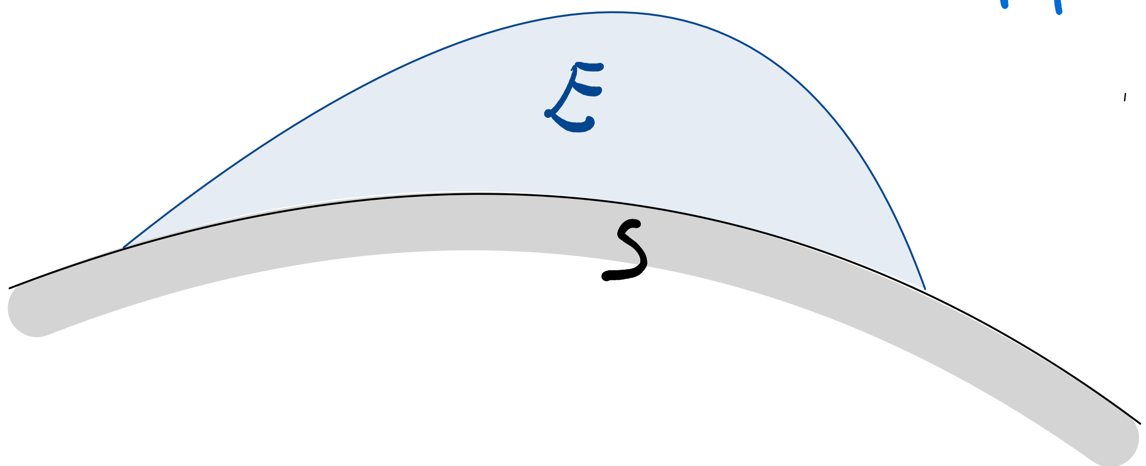
$$\boxed{\mathcal{E}'(E) = \sigma(\partial E \setminus S) - \underbrace{\lambda\,\sigma(\partial S \cap \partial E)}_{\text{!! WETTING ENERGY}}} \qquad |\lambda| < 1$$

Gauss' variational approach



$$\boxed{\mathcal{E}'(E) = \sigma(\partial E \setminus S) - \underbrace{\lambda\, \sigma(\partial S \cap \partial E)}_{\text{!! WETTING ENERGY}} + \underbrace{\int_E g \, dV}_{\text{potential energy (e.g. gravity)}}}$$

Gauss' variational approach

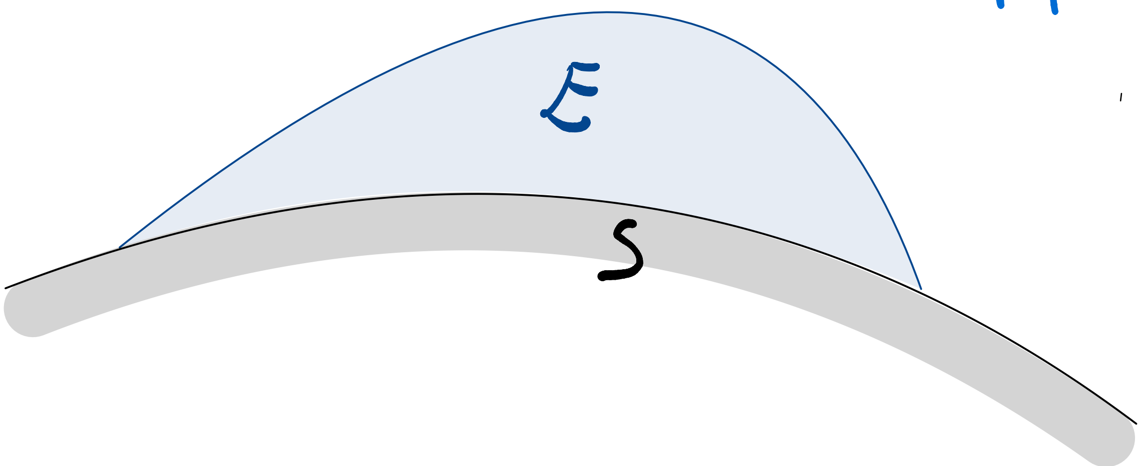


$$\boxed{\mathcal{E}'(E) = \sigma(\partial E \setminus S) - \underbrace{\lambda\,\sigma(\partial S \cap \partial E)}_{\text{WETTING ENERGY}} + \underbrace{\int_E g\, dV}_{\text{potential energy (e.g. gravity)}}}$$

!!

EQUILIBRIUM CONFIGURATIONS: (local) minimizers of $\mathcal{E}'$ under VOLUME CONSTRAINT

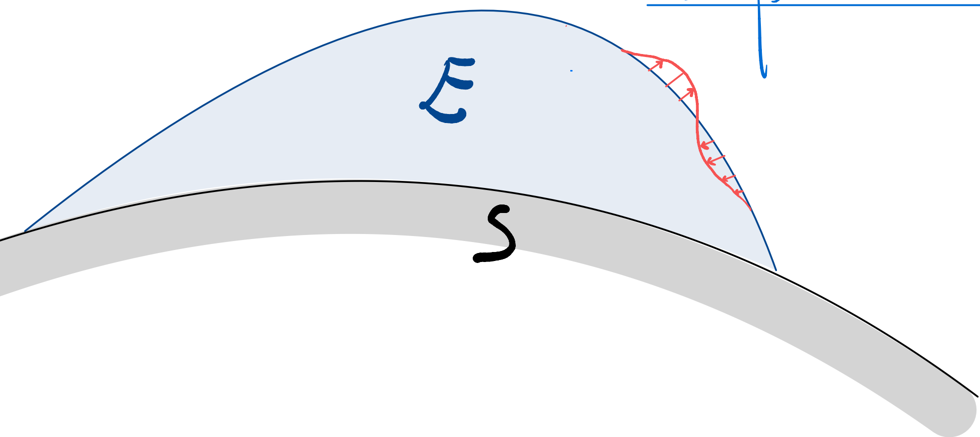
Gauss' variational approach



$$\mathcal{E}'(E) = \sigma(\partial E \setminus S) - \underbrace{\lambda\,\sigma(\partial S \cap \partial E)}_{\text{!! WETTING ENERGY}} + \underbrace{\int_E (g - \Lambda)\,dV}_{\text{potential energy + VOLUME PENALIZATION}}$$

EQUILIBRIUM CONFIGURATIONS: (local) minimizers of $\mathcal{E}'$ ~~under VOLUME CONSTRAINT~~

Equilibrium Conditions



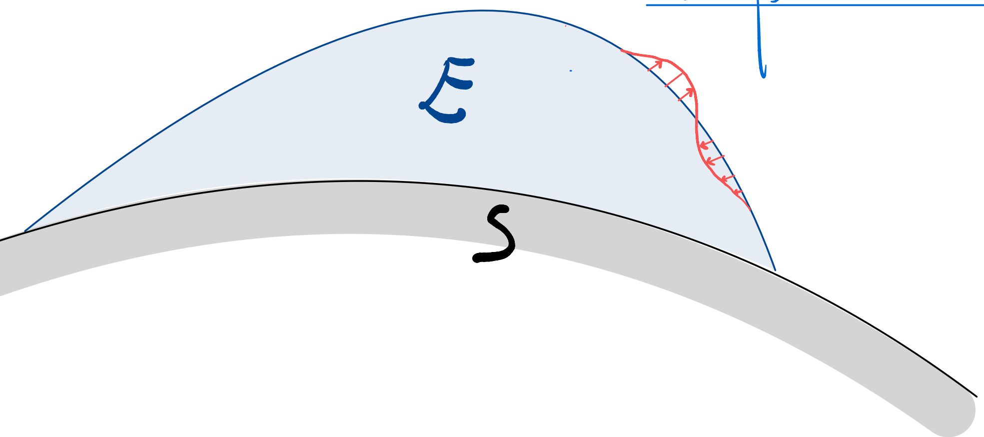
OUTER NORMAL TO ∂E

$$\partial E_\varepsilon \setminus S = \{ x + \varepsilon \varphi(x) \nu_E(x) : x \in \partial E \}$$

$\left(+ \int_{\partial E} \varphi \, d\sigma = 0 \right.$ in case of VOLUME CONSTRAINT$\left. \right)$

$$\frac{d}{d\varepsilon} \mathcal{E}'(E_\varepsilon)\Big|_{\varepsilon=0} = \frac{d}{d\varepsilon}\left(\sigma(\partial E_\varepsilon \setminus S) + \int_{E_\varepsilon} g \, dV \right)\Big|_{\varepsilon=0} = \int_{\partial E} \left(H_{\partial E} + g \right) \varphi \, d\sigma = 0$$

Equilibrium Conditions



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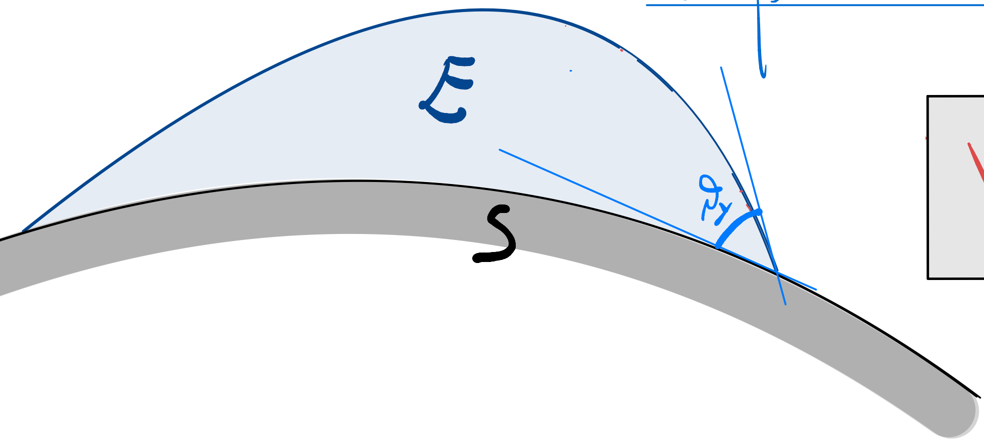
$$\Longrightarrow \quad \boxed{H_{\partial E} + g = \text{CONST}}$$

Equilibrium Conditions

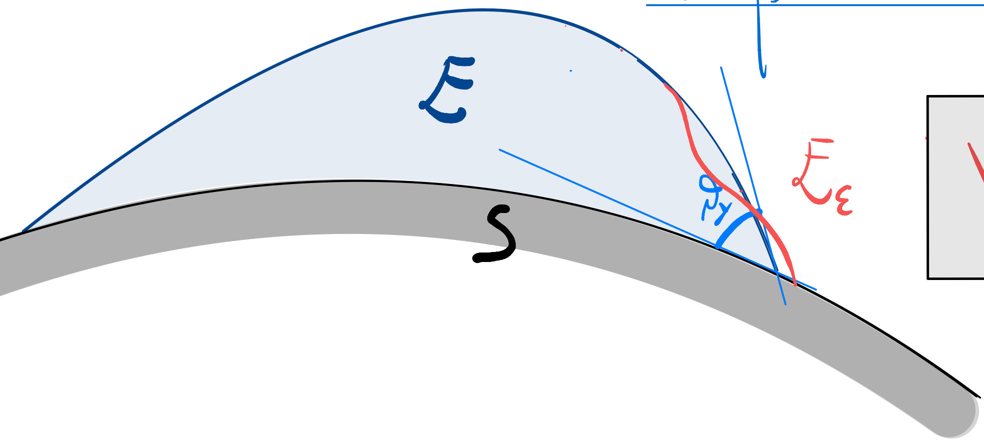


YOUNG'S LAW: $\cos \vartheta_Y = \lambda$

COEFFICIENT of the WETTING ENERGY.

(Recall: $\mathcal{E}(E) = \sigma(\partial E \setminus S) - \lambda\, \sigma(\partial E \cap S) + \int_E g\, dV$)

Equilibrium Conditions



YOUNG'S LAW: $\cos \vartheta_Y = \lambda$

COEFFICIENT of the WETTING ENERGY.

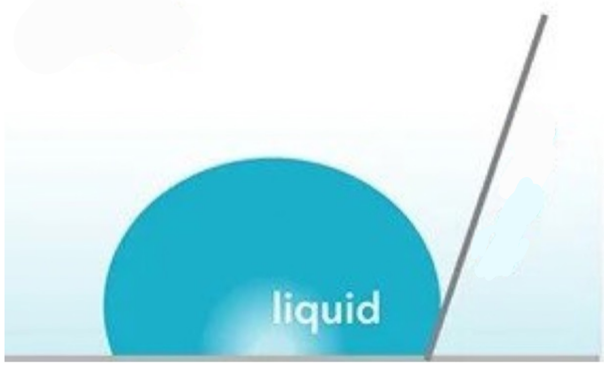
(Recall: $\mathcal{E}'(E) = \sigma(\partial E \setminus S) - \lambda\,\sigma(\partial E \cap S) + \int_E g\, dV$)

$$0 = \frac{d}{d\varepsilon}\, \mathcal{E}'(E_\varepsilon)\Big|_{\varepsilon=0} = \frac{d}{d\varepsilon}\left(\sigma(\partial E_\varepsilon \setminus S) - \lambda\,\sigma(\partial E_\varepsilon \cap S) + \int_{E_\varepsilon} g\, dV\right)\Big|_{\varepsilon=0} \Longrightarrow \cos \vartheta_Y = \lambda$$

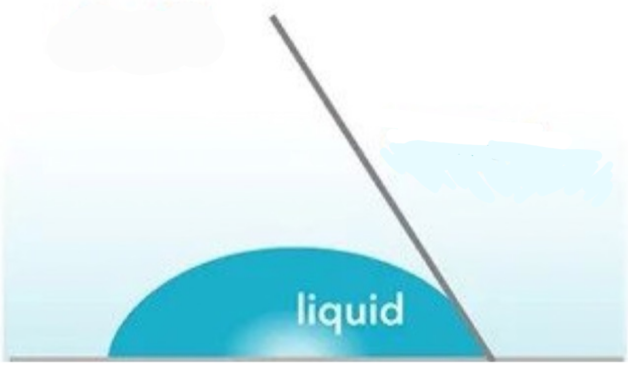
The sessile drop

Set $H := \{x_3 > 0\}$. The SHAPE of the SESSILE DROP (when $g=0$) is described by:

$$\boxed{\min \left\{ \mathcal{E}(E) := \sigma(\partial E \cap H) - \lambda\, \sigma(\partial E \cap \partial H) : E \subseteq H \text{ s.t. } |E| = m \right\}}$$



$\lambda \in (-1,0)$, $\vartheta_Y > \frac{\pi}{2}$

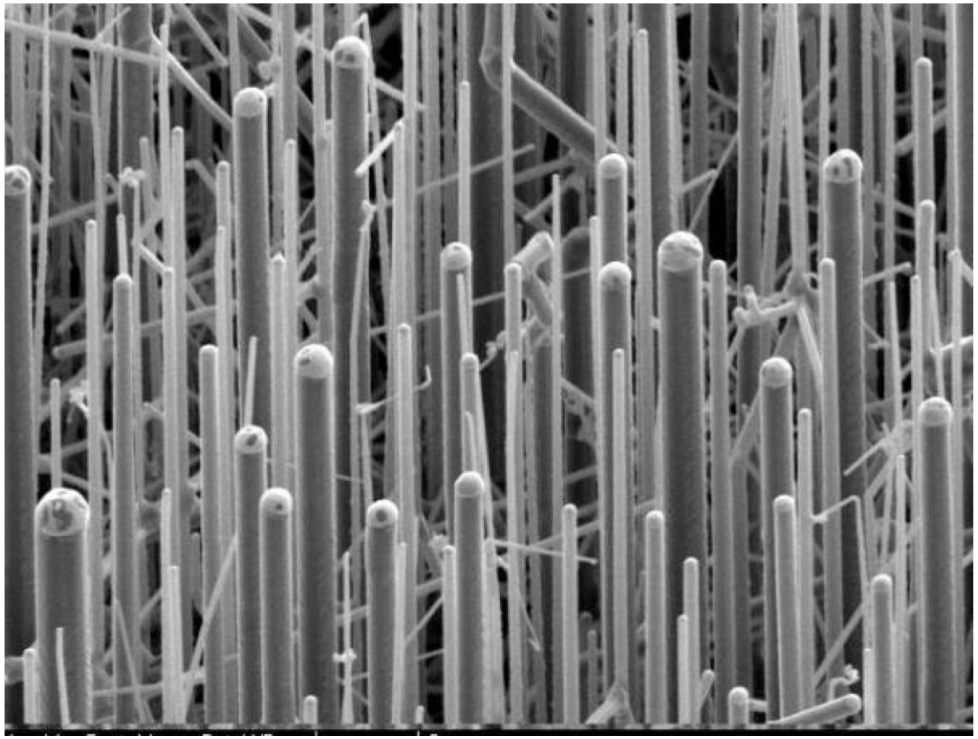


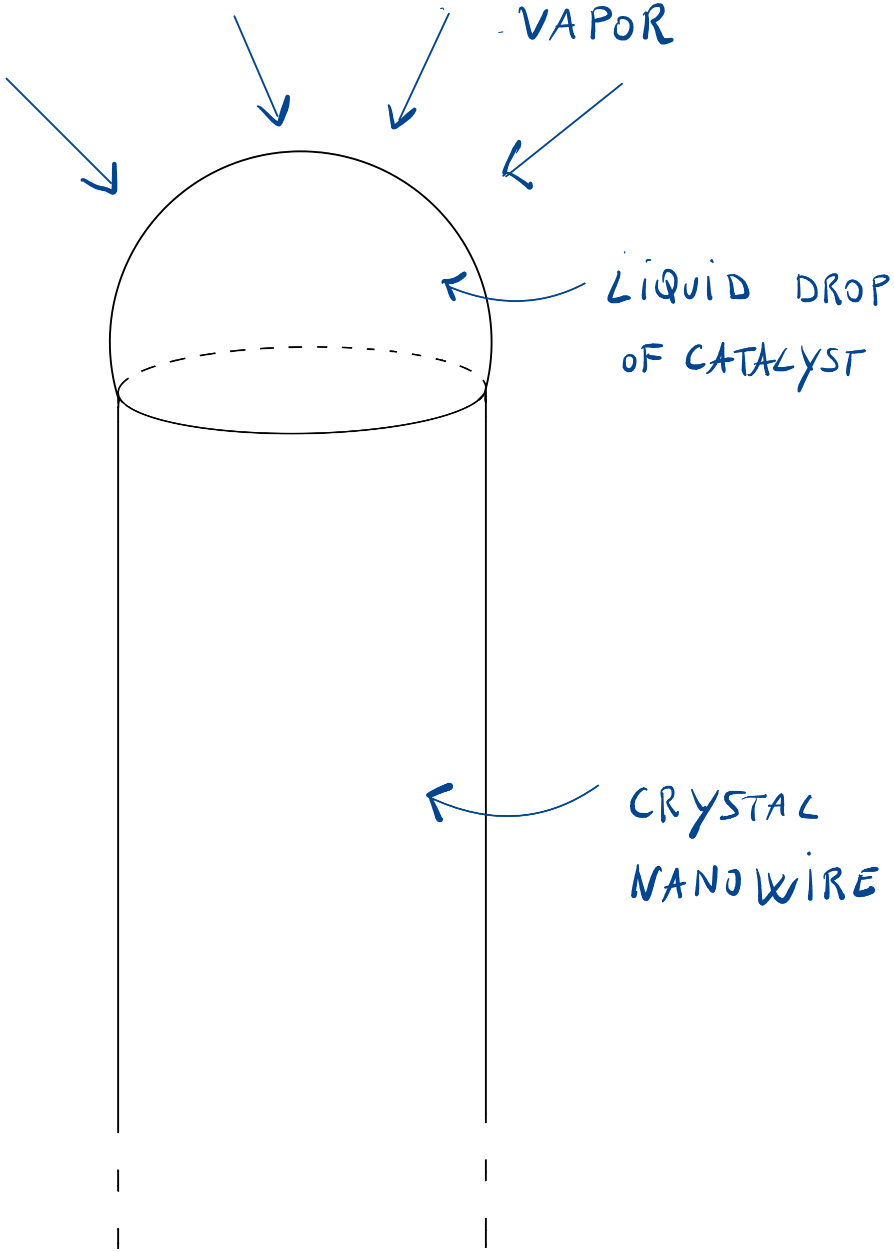
$\lambda \in (0,1)$, $\vartheta_Y < \frac{\pi}{2}$

$\lambda = 0$, $\vartheta_Y = \frac{\pi}{2}$

The SOLUTION is given by the SPHERICAL CAP S_λ touching the support ∂H with YOUNG'S CONTACT ANGLE $\vartheta_Y = \arccos \lambda$

Nanowire growth

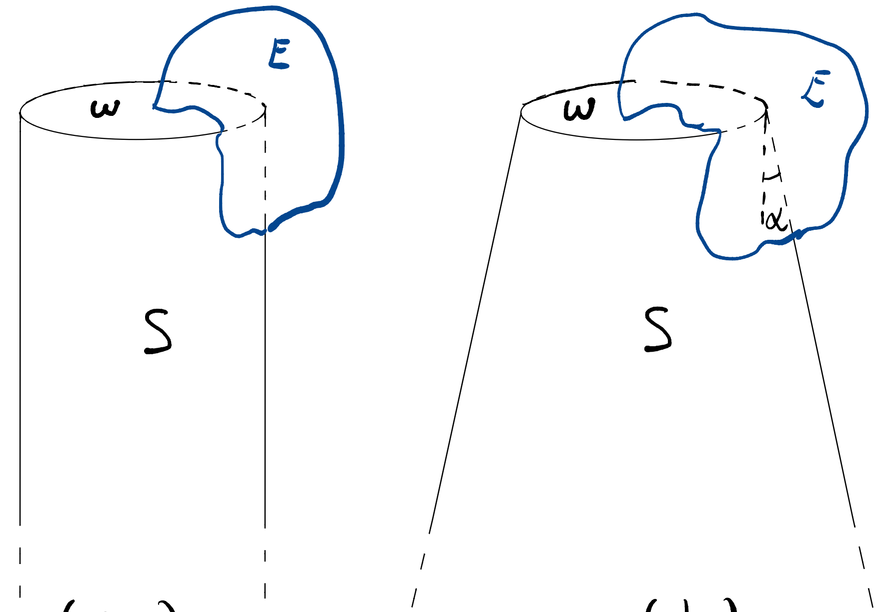




APPLICATIONS

- nanoelectronic devices
- biological applications
- battery electrodes

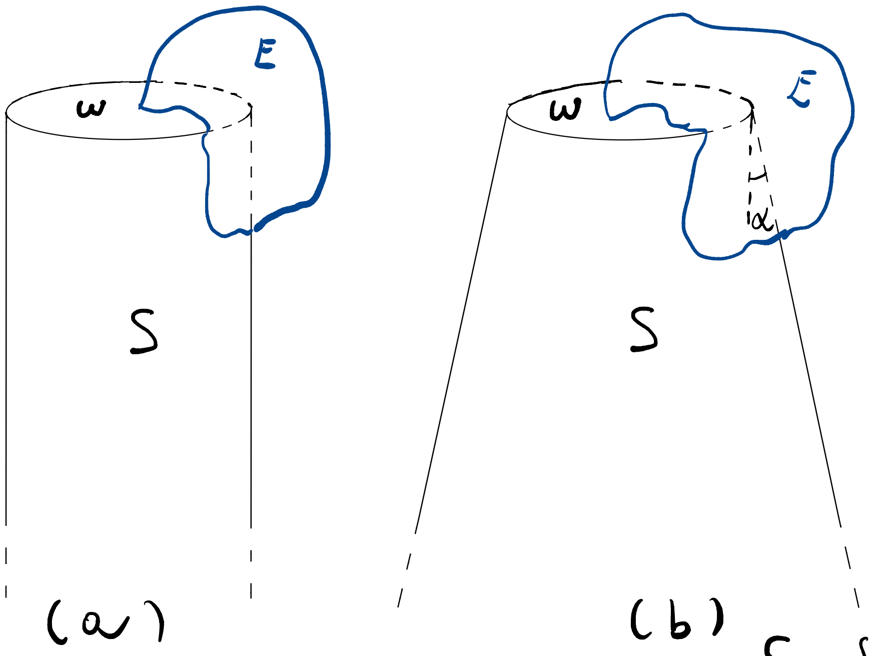
Nanowire growth - II



(a) (b)

- GRAVITY NEGLECTED (small mass regime)

Nanowire growth - II



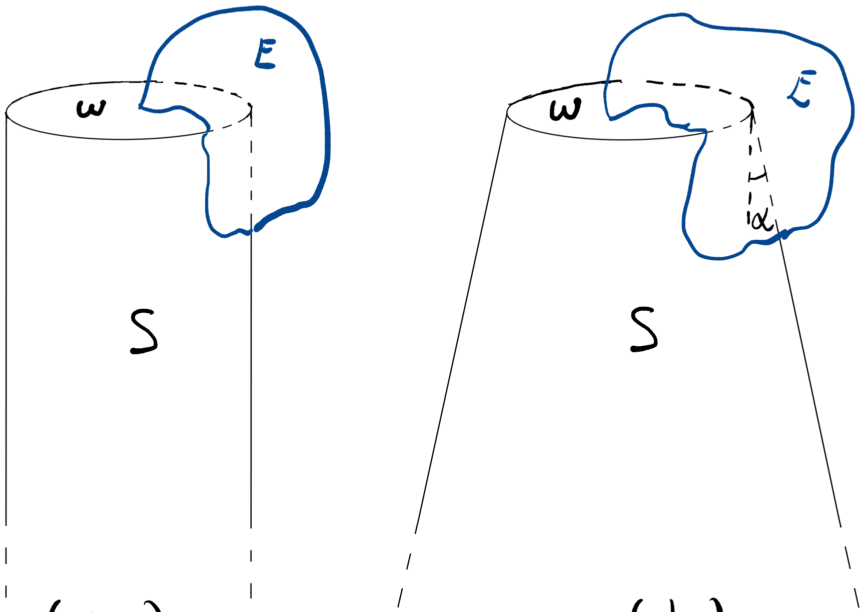
- GRAVITY NEGLECTED (small mass regime)
- EQUILIBRIUM CONFIGURATIONS:

(LOCAL) MINIMIZERS of

$$\boxed{(P)\ \min\left\{\sigma(\partial^* E\setminus S)-\lambda\,\sigma(\partial^* E\cap S):\ |E|=m\right\}}$$

$$S=\left\{(x',t)\in\mathbb{R}^2\times(-\infty,0]:\ x'\in(1-t\tan\alpha)\,\omega\right\},\quad \underset{\text{CONVEX}}{\omega\subseteq\mathbb{R}^2}$$

Nanowire growth - II



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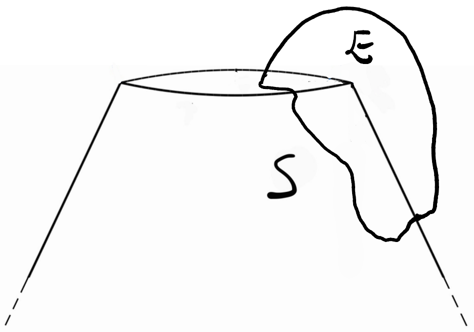
THEOREM (FONSECA - FUSCO - LEONI - M. '22)

In case (a) problem (P) admits a GLOBAL MINIMIZER $\forall\lambda\in(-1,1)$.

If $\lambda=0$ there is NO GLOBAL MINIMIZER in case (b) if $m>m_0$.

Non existence in case (b)



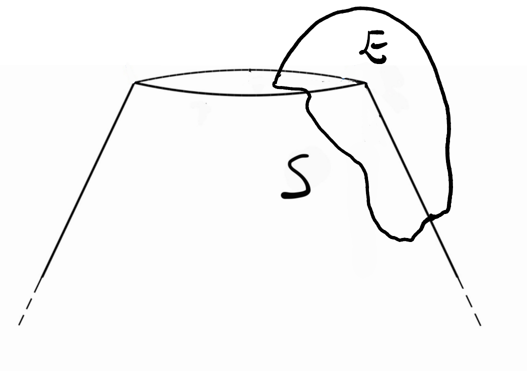
$$\min \left\{ \sigma(\partial^* E \setminus S) : E \subseteq \mathbb{R}^3 \setminus S, \ |E| = m \right\}$$

Theorem (CHOE–GHOMI–RITORÉ '06, FUSCO–M. '21) S CLOSED CONVEX BODY. Then $\forall E \subseteq \mathbb{R}^3 \setminus S$ of FINITE PERIMETER

$$\sigma(\partial^* E \setminus S) \geq 3 \left(\frac{2}{3} \pi \right)^{\frac{1}{3}} |E|^{\frac{2}{3}},$$

with EQUALITY iff E is HALF BALL sitting on a FACET of S.

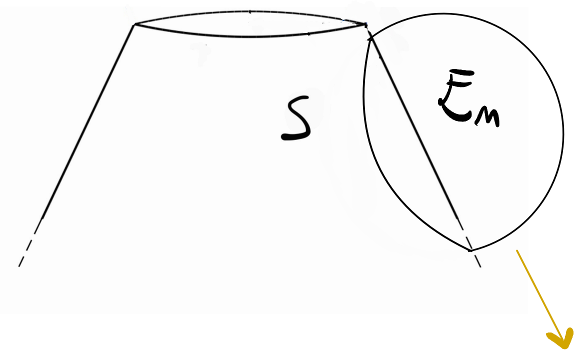
Non existence in case (b)



$$\min \left\{ \sigma(\partial^* E \setminus S) : E \subseteq \mathbb{R}^3 \setminus S,\ |E| = m \right\}$$

$$\boxed{\sigma(\partial^* E \setminus S) > 3 \left(\tfrac{2}{3}\pi\right)^{\frac{1}{3}} |m|^{\frac{2}{3}} \qquad \forall E \text{ admissible}}$$

Non existence in case (b)



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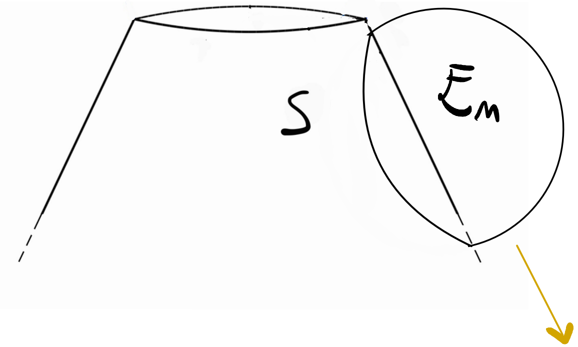

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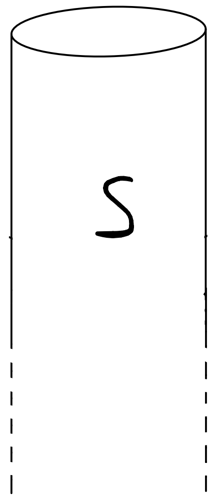
$$\boxed{\sigma(\partial^* E \setminus S) > 3 \left(\tfrac{2}{3}\pi\right)^{\frac{1}{3}} |m|^{\frac{2}{3}} \qquad \forall E \text{ admissible}}$$

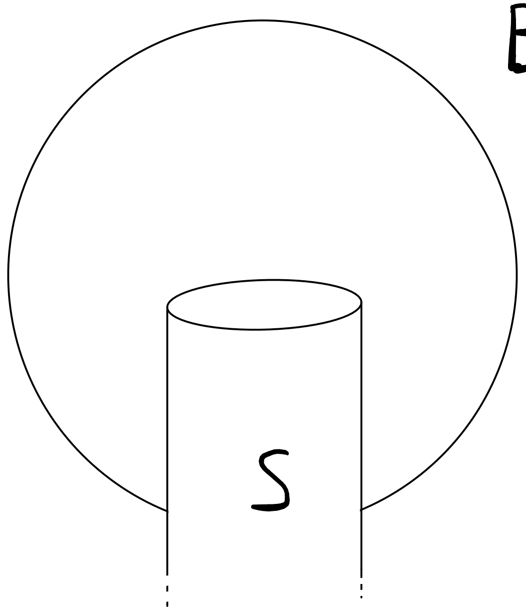
$$\sigma(\partial^* E_m \setminus S) \searrow 3 \left(\tfrac{2}{3}\pi\right)^{\frac{1}{3}} |m|^{\frac{2}{3}} \rightsquigarrow \text{NO MINIMIZER}$$

Existence in case (a)



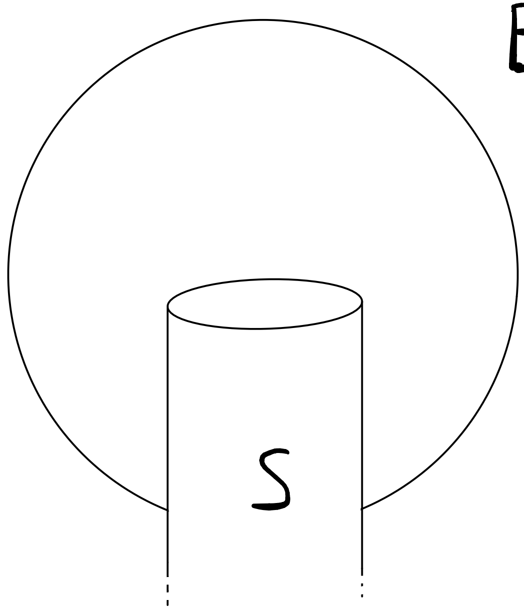
Existence in case (a)

$B_{R_n}, \quad R_n \to +\infty$



- $E_n^\Lambda \in \operatorname{argmin}\left\{ \sigma(\partial^* E \setminus S) - \lambda\, \sigma(\partial^* E \cap S) + \Lambda \big|\, |E| - m \,\big| : E \subseteq B_{R_n} \setminus S \right\}$

Existence in case (a)

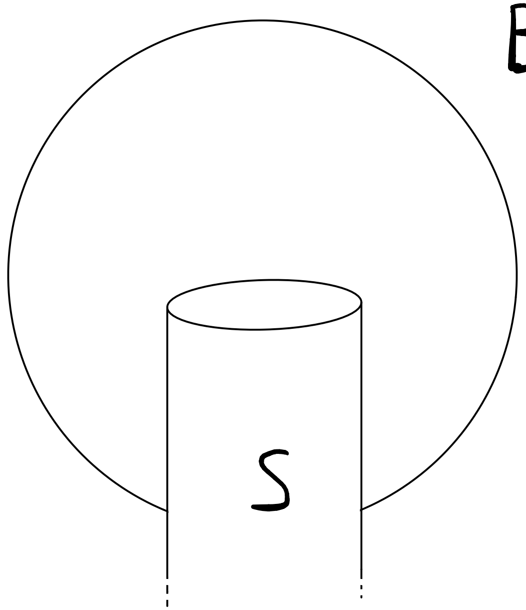


B_{R_n} , $R_n \longrightarrow +\infty$

- $E_n^\Lambda \in \operatorname{argmin} \left\{ \sigma(\partial^* E \setminus S) - \lambda\, \sigma(\partial^* E \cap S) + \Lambda \big| |E| - m \big| : E \subseteq B_{R_n} \setminus S \right\}$
- VOLUME DENSITY ESTIMATES $\Longrightarrow$ # CONNECTED COMPONENTS $E_{n,i}^\Lambda$ of $E_n^\Lambda \leq C(\Lambda)$
  $\operatorname{diam}(E_{n,i}^\Lambda) \leq C(\Lambda)$

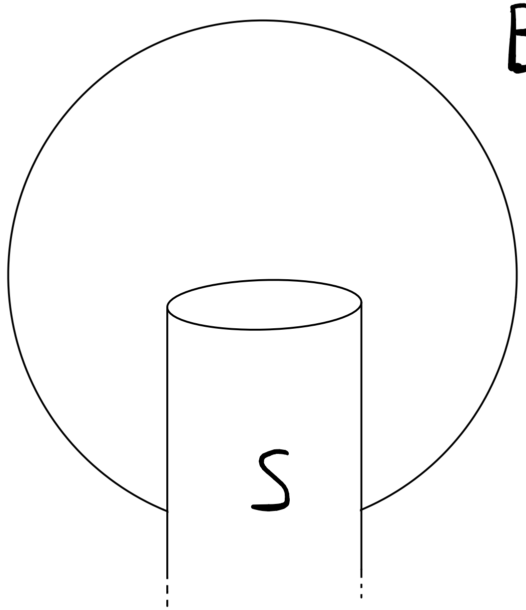
Existence in case (e)

B_{R_m}, $R_m \longrightarrow +\infty$



- $E_m^\Lambda \in \operatorname{argmin}\{\sigma(\partial^* E \setminus S) - \lambda\,\sigma(\partial^* E \cap S) + \Lambda\,\big||E| - m\big| : E \subseteq B_{R_m} \setminus S\}$
- VOLUME DENSITY ESTIMATES $\Rightarrow$ # CONNECTED COMPONENTS $E_{m,i}^\Lambda$ of $E_m^\Lambda \leq C(\lambda)$
  $\operatorname{diam}(E_{m,i}^\Lambda) \leq C(\lambda)$
- By SLIDING VERTICALLY $E_{m,i}^\Lambda$ we may enforce $\{E_m^\Lambda\}_m$ EQUI-BOUNDED

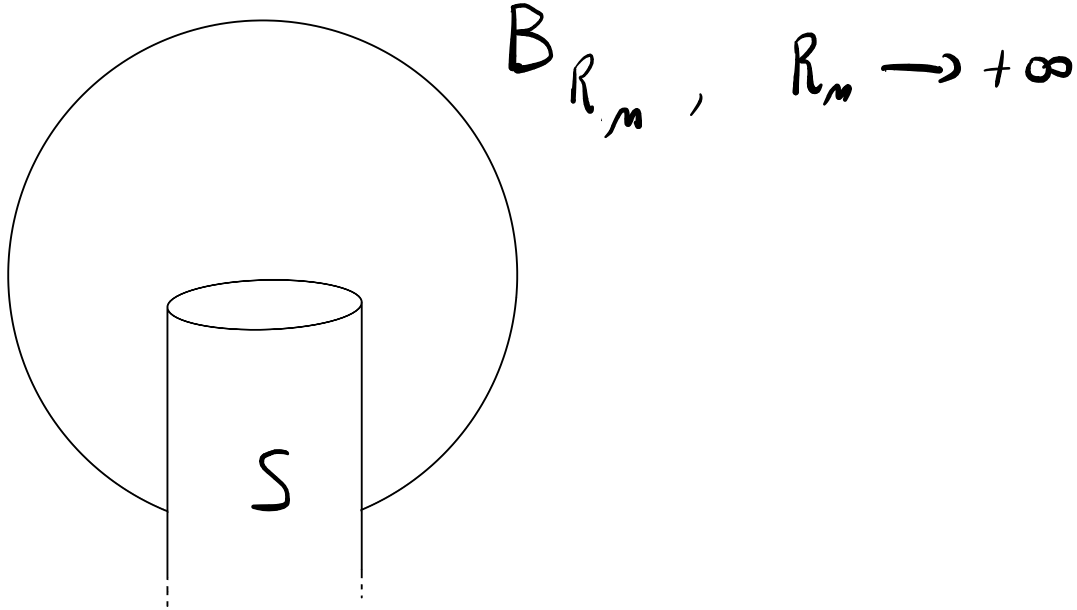
Existence in case (e)



B_{R_n}, $R_n \to +\infty$

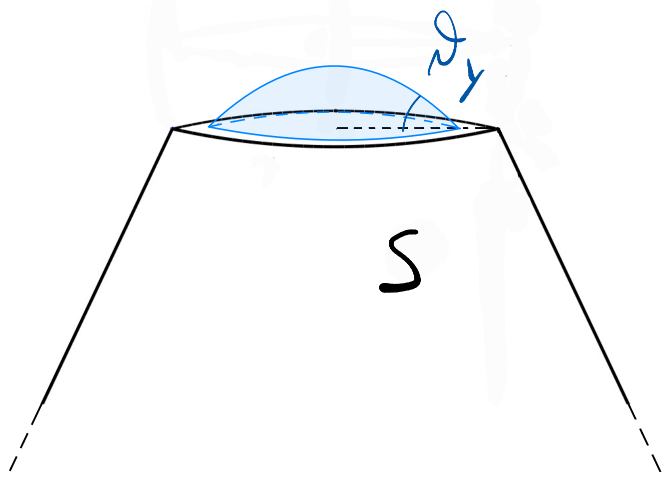
- $E_n^\Lambda \in \operatorname{argmin} \left\{ \sigma(\partial^* E \setminus S) - \lambda \sigma(\partial^* E \cap S) + \Lambda \big| |E| - m \big| : E \subseteq B_{R_n} \setminus S \right\}$
- VOLUME DENSITY ESTIMATES $\Rightarrow$ # CONNECTED COMPONENTS $E_{n,i}^\Lambda$ of $E_n^\Lambda \leq C(\Lambda)$
  $\operatorname{diam}(E_{n,i}^\Lambda) \leq C(\Lambda)$
- By SLIDING VERTICALLY $E_{n,i}^\Lambda$ we may enforce $\{E_n^\Lambda\}_n$ EQUI-BOUNDED
  $\Rightarrow E_n^\Lambda \to E^\Lambda$, with E^Λ a minimizer of the PENALIZED PROBLEM

Existence in case (a)



- $E_n^\Lambda \in \operatorname{argmin}\{\sigma(\partial^* E \setminus S) - \lambda\,\sigma(\partial^* E \cap S) + \Lambda\,||E| - m| : E \subseteq B_{R_n} \setminus S\}$
- VOLUME DENSITY ESTIMATES $\Rightarrow$ # CONNECTED COMPONENTS $E^\Lambda_{n,i}$ of $E_n^\Lambda \leq C(\Lambda)$
  $\operatorname{diam}(E^\Lambda_{n,i}) \leq C(\Lambda)$
- By SLIDING VERTICALLY $E^\Lambda_{n,i}$ we may enforce $\{E_n^\Lambda\}_n$ EQUI-BOUNDED
  $\Rightarrow E_n^\Lambda \to E^\Lambda$, with E^Λ a minimizer of the PENALIZED PROBLEM
- Λ is LARGE ENOUGH $\Rightarrow |E^\Lambda| = m$, hence a MINIMIZER.

The pinning effect



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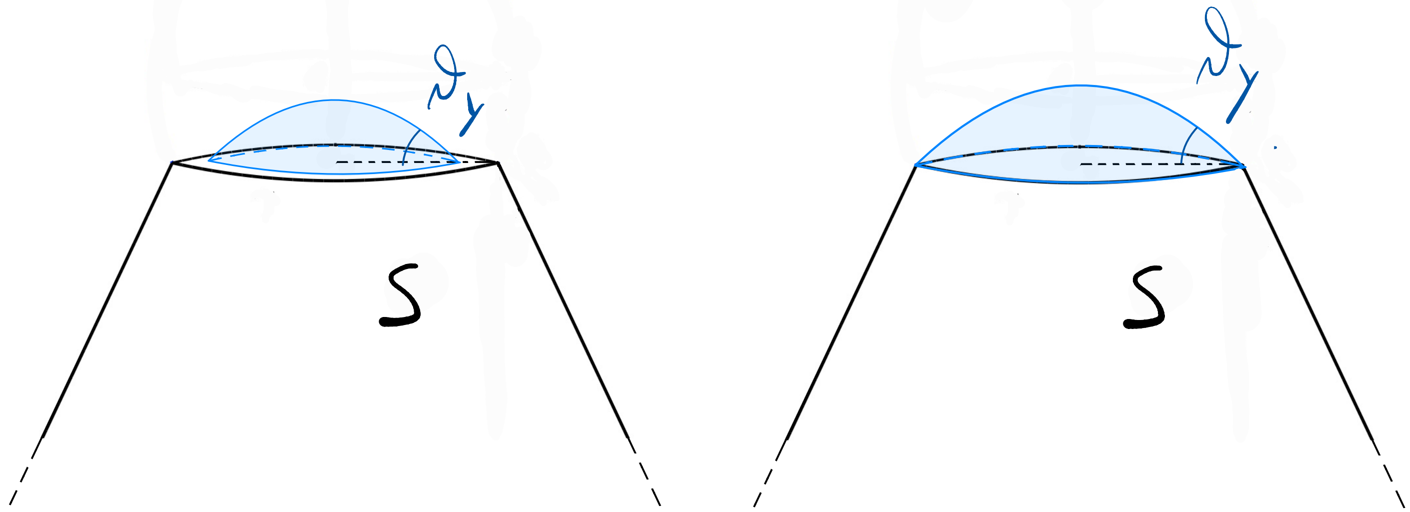
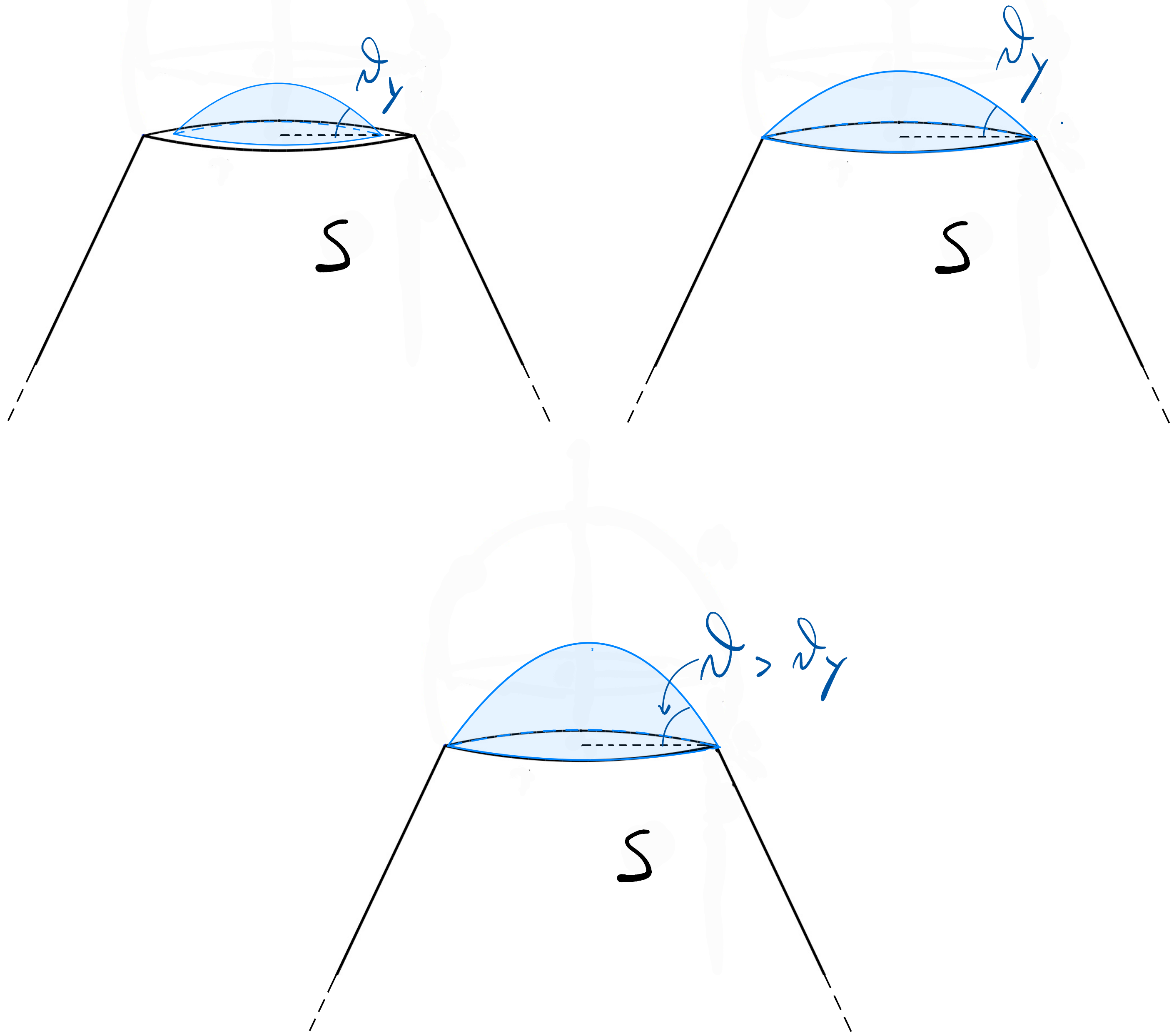
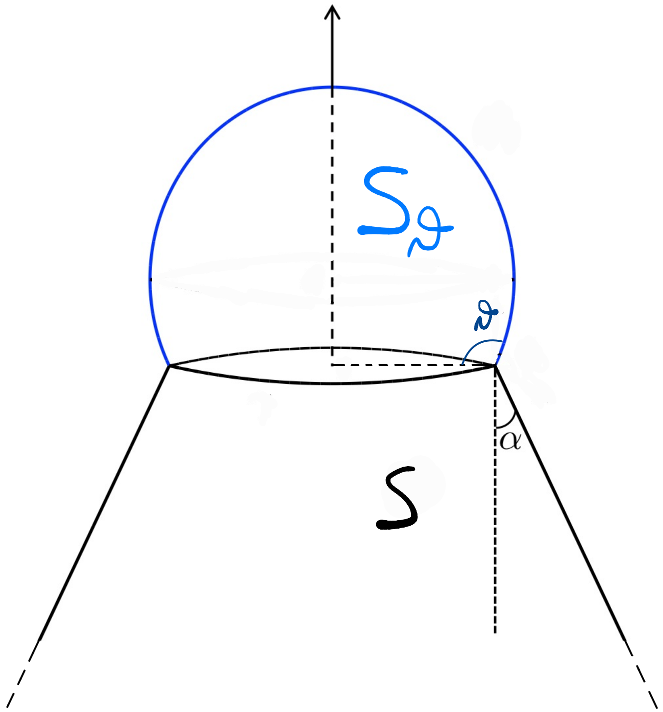


The pinning effect

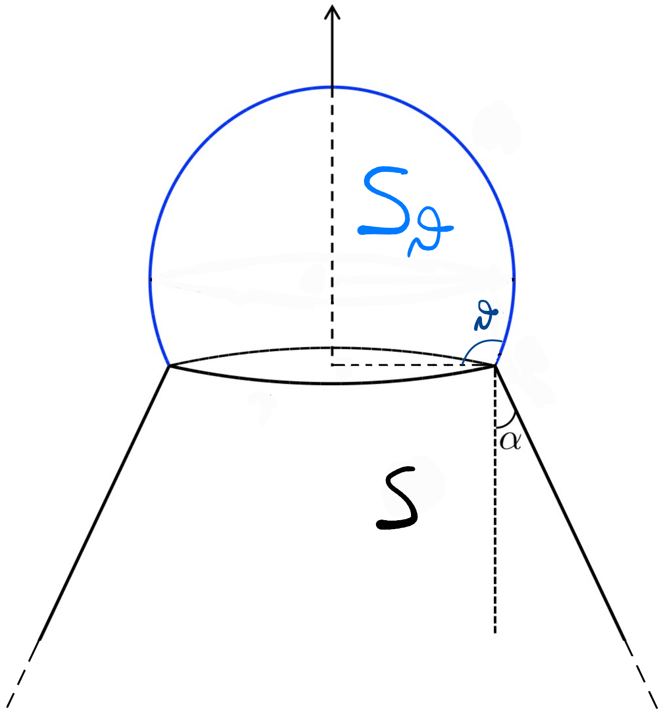


Axisymmetric Nanowires



- $\vartheta > \vartheta_Y = \arccos\lambda$, $\alpha \in [0, \frac{\pi}{2})$.

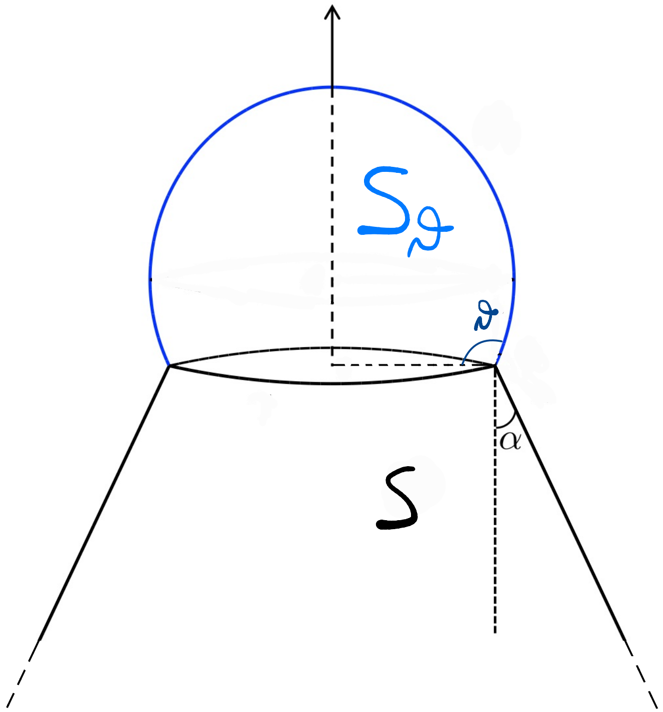
Axisymmetric Nanowires



- $\vartheta > \vartheta_Y = \arccos\lambda$, $\alpha \in [0, \frac{\pi}{2})$.
- Find conditions under which S_ϑ is a local minimizer of

$$\boxed{(\mathcal{P}) \min\left\{ \sigma(\partial^* E \setminus S) - \lambda\,\sigma(\partial^* E \cap S) : |E| = |S_\vartheta| \right\}}$$

Axisymmetric Nanowires



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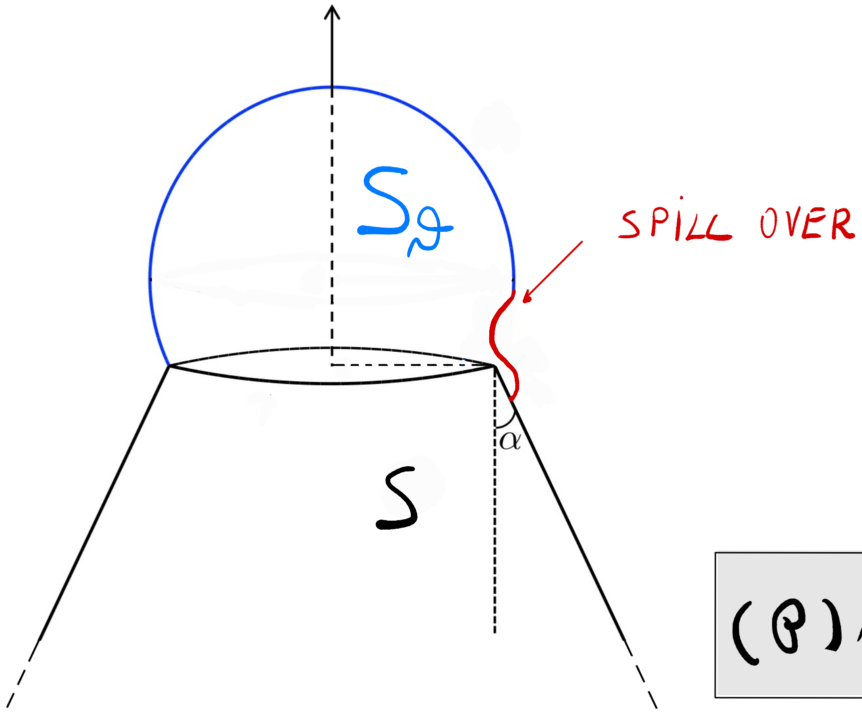
THEOREM (FONSECA - FUSCO - LEONI - M. '22)

- If $\vartheta_Y < \vartheta \leq \frac{\pi}{2} - \alpha - \vartheta_Y$, then S_ϑ is a LOCAL MINIMIZER
- If $\vartheta > \frac{\pi}{2} - \alpha - \vartheta_Y$, then SPILL OVER occurs.

Axisymmetric Nanowires



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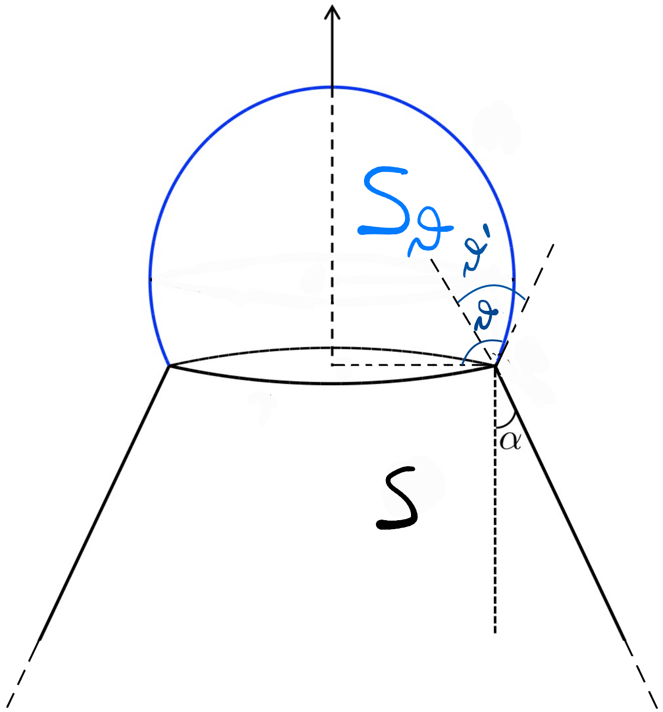

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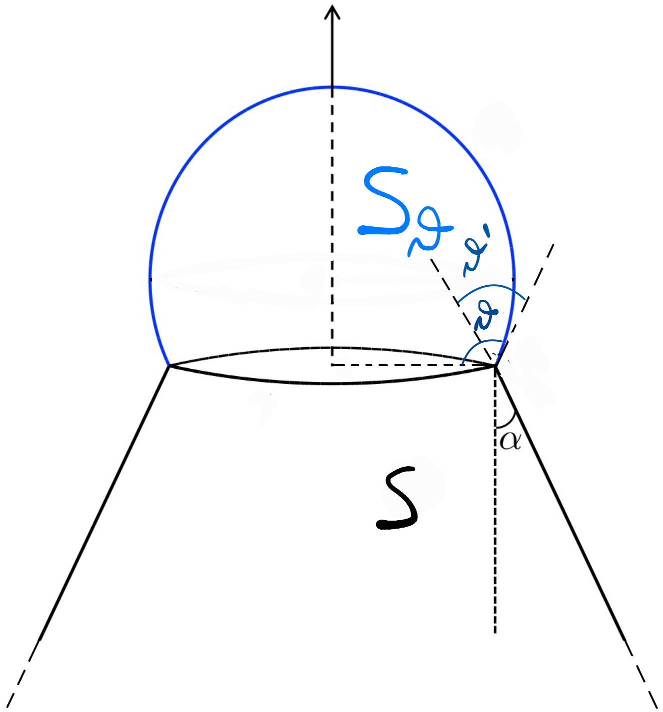
Rmk: YOUNG'S LAW is violated due to SHARP EDGE

Axisymmetric Nanowires



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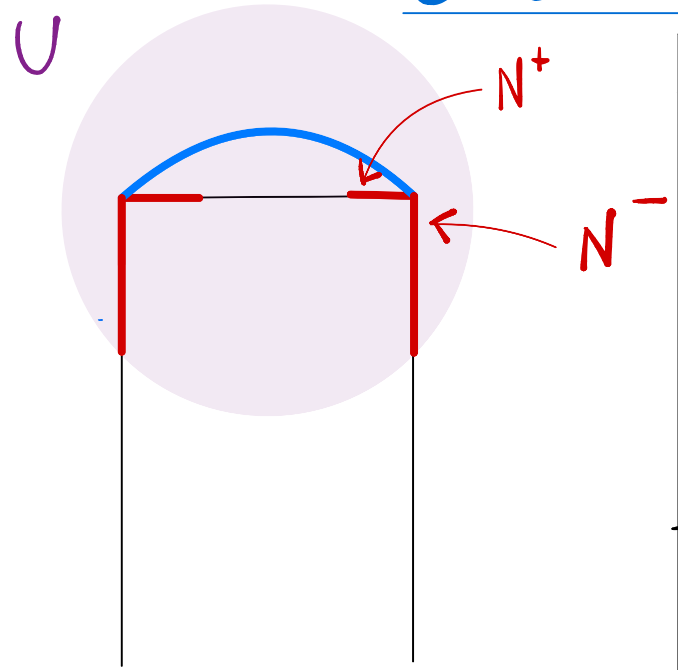
- If $\vartheta_Y < \vartheta \leq \frac{\pi}{2} - \alpha - \vartheta_Y$, then S_ϑ is a LOCAL MINIMIZER
- If $\vartheta > \frac{\pi}{2} - \alpha - \vartheta_Y$, then SPILL OVER occurs.

$\rightsquigarrow$ analytical validation of OLIVER-HUH-MASON '77

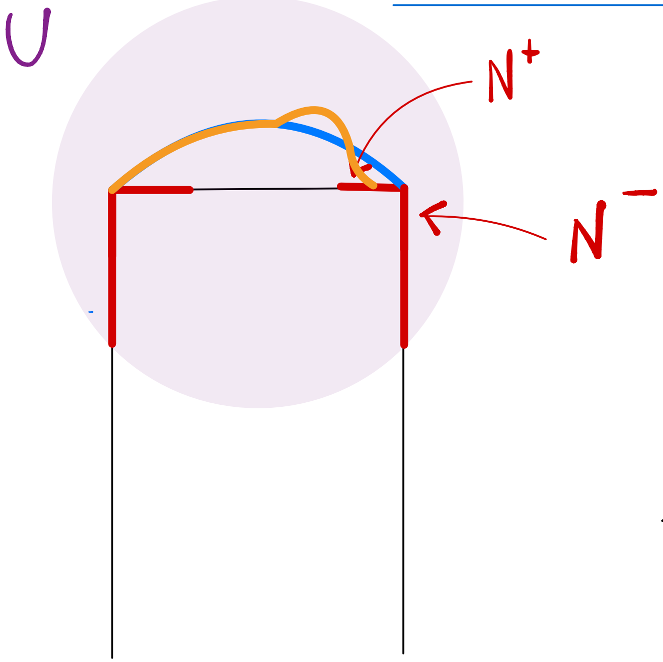
The calibration method



- $\xi : U \to \mathbb{R}^3$ s.t.
  - $\operatorname{div} \xi \equiv \text{cost}$
  - $\xi = \nu_{S_\vartheta}$ on $\partial S_\vartheta \setminus S$
  - $|\xi| \leq 1$
  - $\xi \cdot \nu_S \leq \lambda$ on N^+, $\xi \cdot \nu_S \geq \lambda$ on N^-

Then $\mathcal{E}_\lambda(E) \geq \mathcal{E}_\lambda(S_\vartheta)$ $\forall E$ s.t. $|E| = |S_\vartheta|$, $E \Delta S_\vartheta \subseteq U$, $(\partial S_\vartheta \cap S) \setminus \partial E \subseteq N^+$

The calibration method



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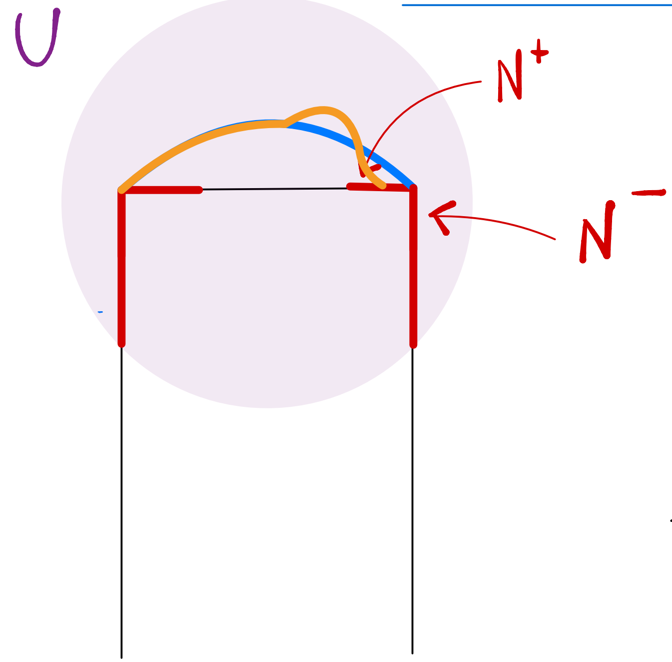
The calibration method



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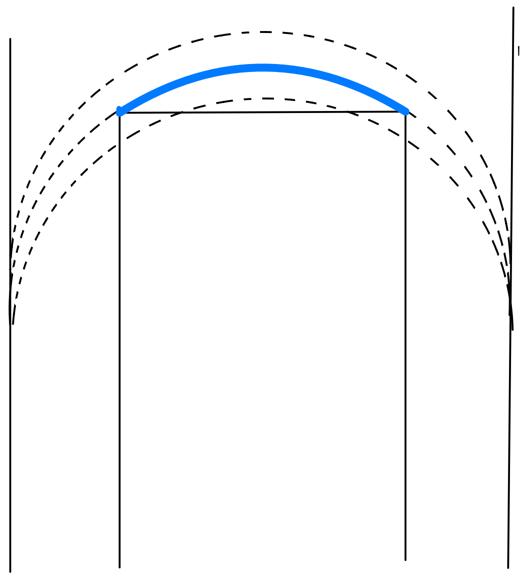
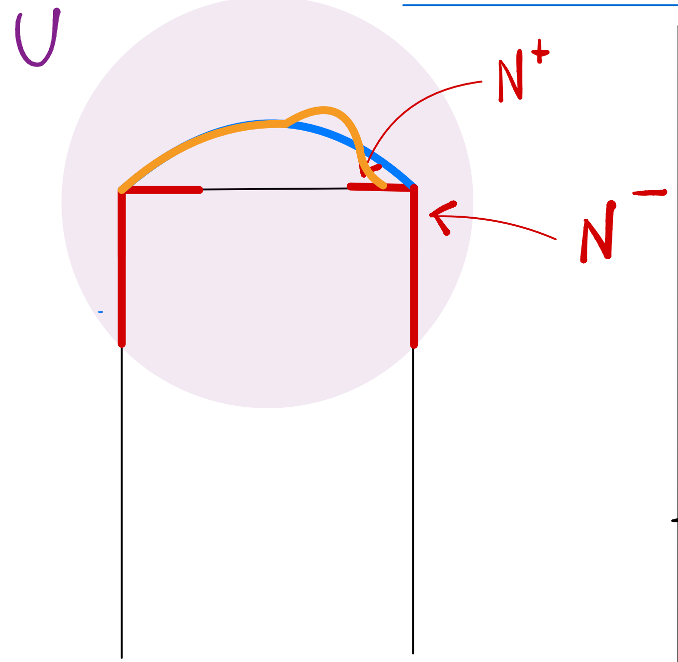
- CONSTRUCTION BY FOLIATION with CMC SURFACES



The calibration method



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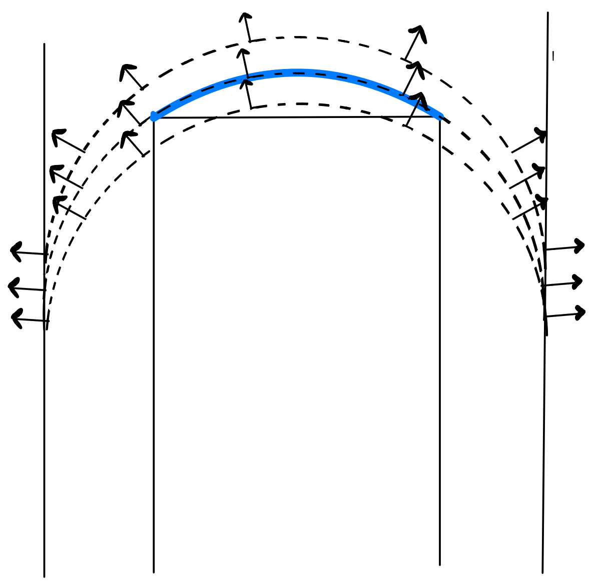
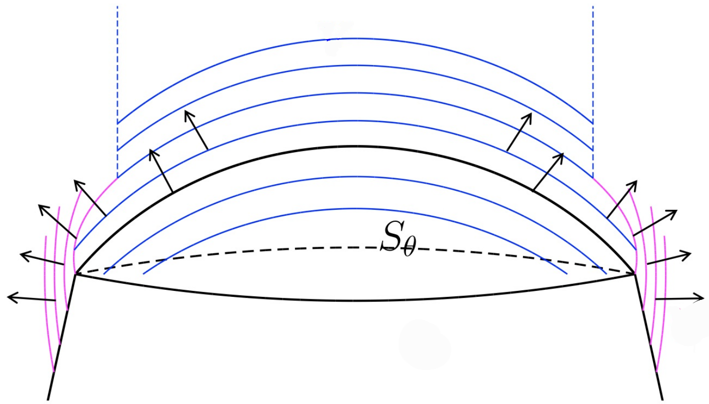
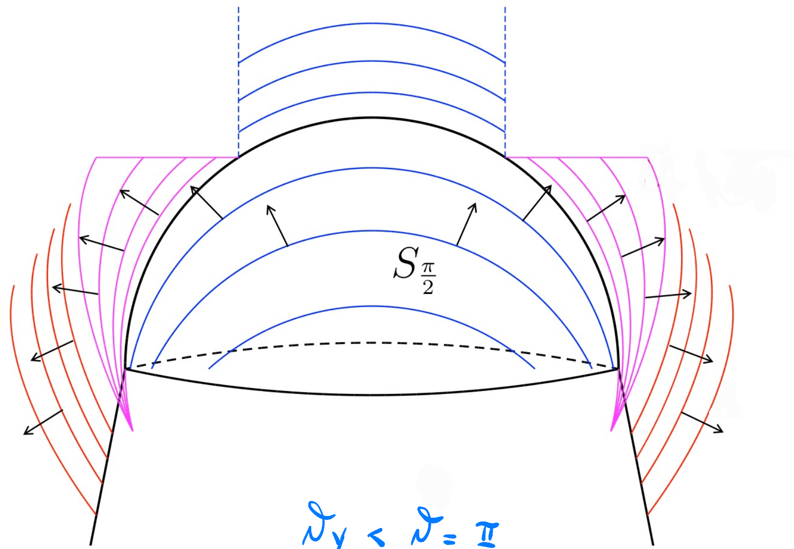
- CONSTRUCTION BY FOLIATION with CMC SURFACES



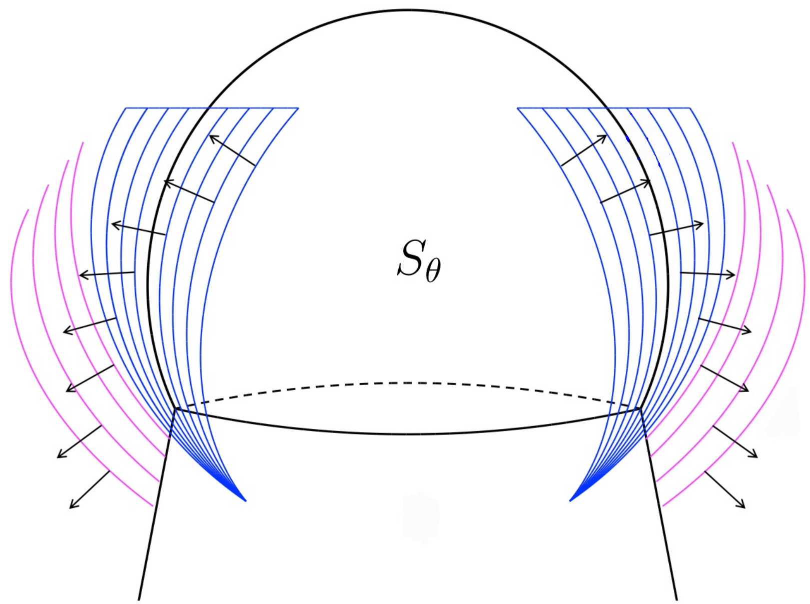
The calibration method – II



$\vartheta_y < \vartheta < \frac{\pi}{2}$

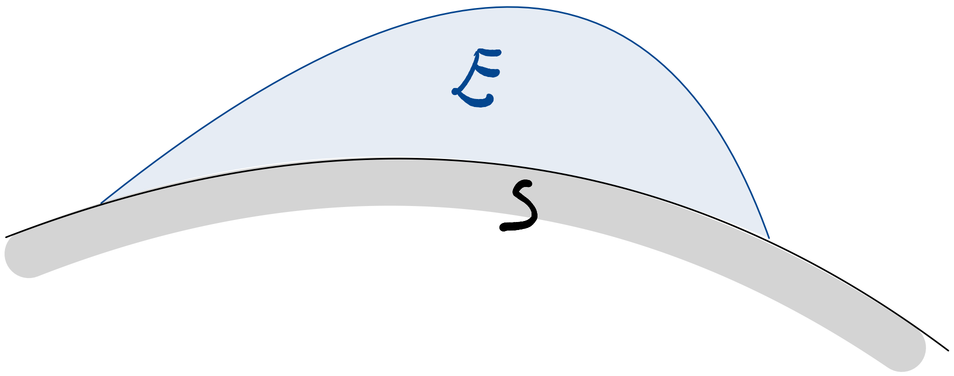


$\vartheta_y < \vartheta = \frac{\pi}{2}$



$\vartheta > \max\{\vartheta_y, \frac{\pi}{2}\}$

Regularity



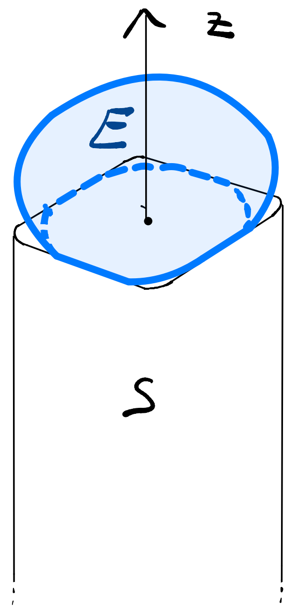
Theorem (TAYLOR '77) Let $E \subseteq \mathbb{R}^3 \setminus S$ be a (LOCAL) minimizer of

$$\min \left\{ \sigma(\partial^* F \setminus S) - \lambda\, \sigma(\partial^* F \cap S) + \int_F g \ : \ F \subseteq \mathbb{R}^3 \setminus S, \ |F| = m \right\}$$

If S is of class C^2 and g is BOUNDED, then ∂E is a SURFACE with BOUNDARY of class $C^{1,\alpha}$ for all $\alpha \in (0,1)$

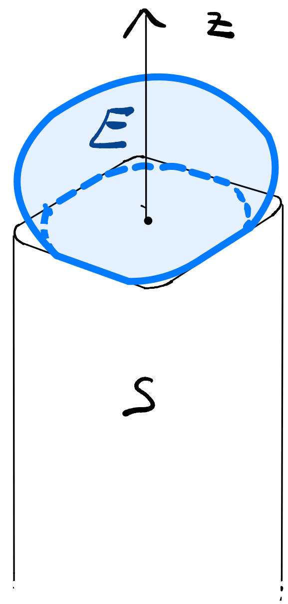
see also DE PHILIPPIS – MAGGI '15

Regularity



Regularity

$H = \{z > 0\}$

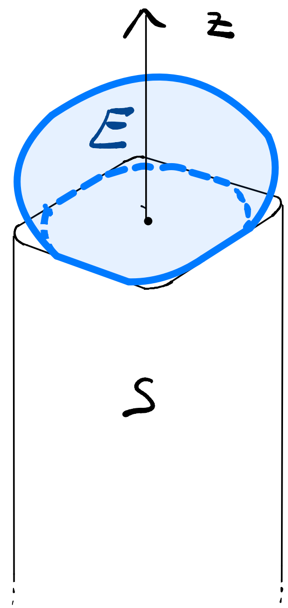


E minimizes:

$$\min\left\{ \sigma(\partial^* F \setminus S) - \lambda\, \sigma(\partial^* F \cap S) : F \subseteq H,\ \partial^* F \cap \partial H \subseteq \omega,\ |F| = m \right\} \quad (\mathcal{P})$$

Regularity

$H = \{z > 0\}$



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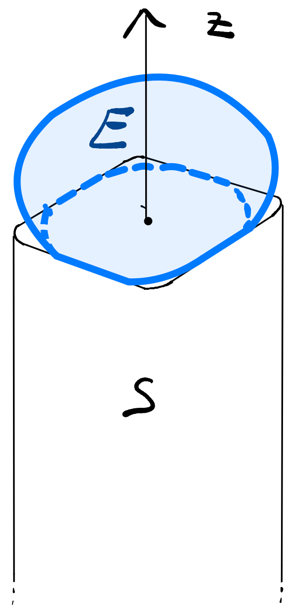
<u>THEOREM</u> (De Philippis – Fusco – M. '22) Let E be a (local) minimizer of $(\mathcal{P})$, and assume $\omega \subseteq \partial H \simeq \mathbb{R}^2$ of class $C^{1,1}$. Then:

Regularity

$H = \{z > 0\}$



E minimizes:

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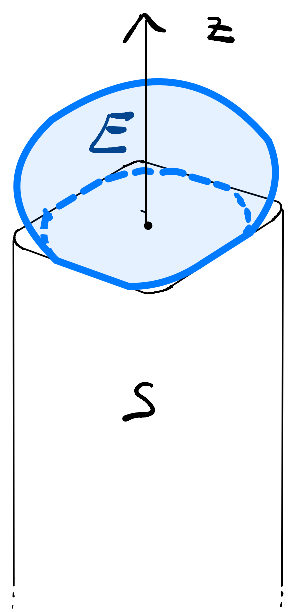
<u>THEOREM</u> (DE PHILIPPIS – FUSCO – M. '22) Let E be a (LOCAL) MINIMIZER of $(\mathcal{P})$, and assume $\omega \subseteq \partial H \simeq \mathbb{R}^2$ of class $C^{1,1}$. Then:

- $\overline{\partial E \cap H}$ is a SURFACE with BOUNDARY of class $C^{1,\gamma}$ $\forall \gamma \in (0, \tfrac{1}{2})$;

Regularity

$H = \{z > 0\}$



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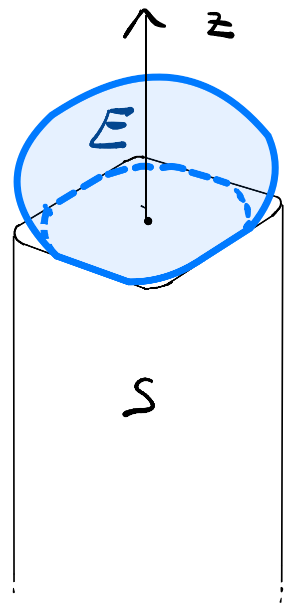
- $\overline{\partial E \cap H}$ is a SURFACE with BOUNDARY of class $C^{1,\gamma}$ $\forall \gamma \in (0, \frac{1}{2})$;
- $\nu_E \cdot e_z = \lambda$ on $\ell \cap \mathring{\omega}$, $\nu_E \cdot e_z \geq \lambda$ on $\ell \cap \partial \omega$

  YOUNG'S LAW   YOUNG'S INEQUALITY

where $\ell := \overline{\partial E \cap H} \cap \partial H$ is the CONTACT LINE

ε-Regularity

$H = \{z > 0\}$



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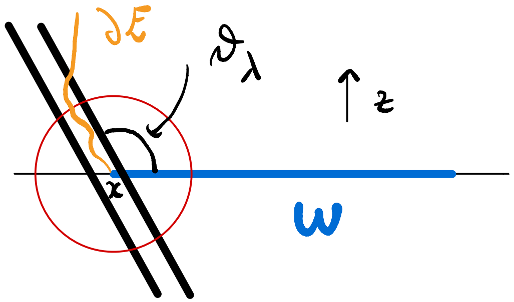

ε-Regularity

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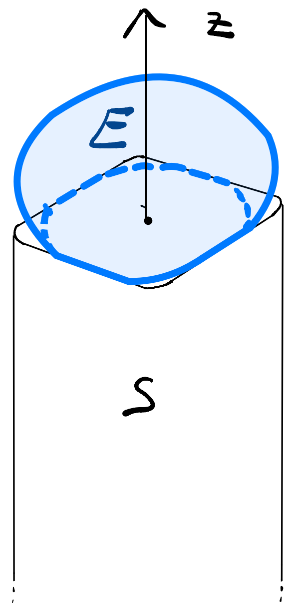

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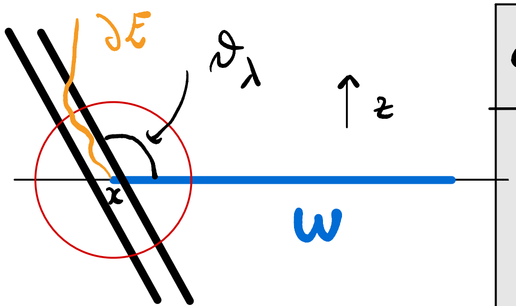


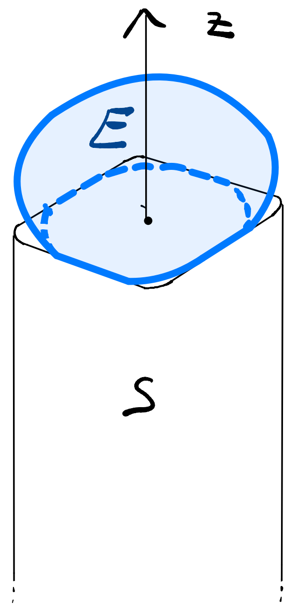
ε-REGULARITY THM (DE PHILIPPIS – FUSCO – P.) Let $E \subseteq H$ be a (LOCAL) MINIMIZER of (P). There exists $\hat{\varepsilon}, \hat{r} > 0$ s.t. if for $x \in \partial\omega \cap \ell$ and $r \leq \hat{r}$

$$\partial E \cap H \cap B_r(x) \subseteq \left\{ \frac{-\lambda z}{\sqrt{1-\lambda^2}} - \hat{\varepsilon} r < x < \frac{-\lambda z}{\sqrt{1-\lambda^2}} + \hat{\varepsilon} r \right\}$$

then $\overline{\partial E \cap H} \cap B_{\frac{r}{2}}(x)$ is a SURFACE with BDRY of class $C^{1,\gamma}$.

ε - Regularity

$H = \{z > 0\}$



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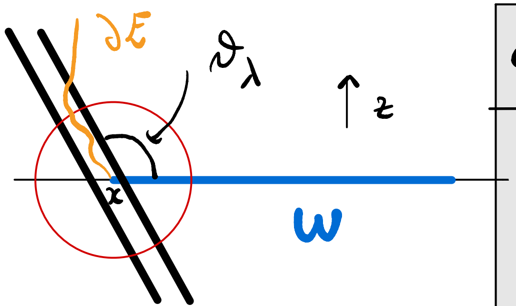

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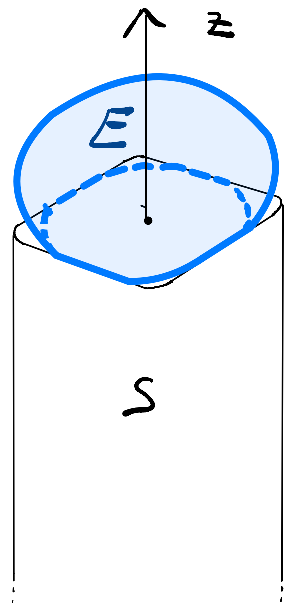
then $\overline{\partial E \cap H} \cap B_{\frac{r}{2}}(x)$ is a SURFACE with BDRY of class $C^{1,\gamma}$.

- ε - REGULARITY HOLDS in ANY DIMENSION

ε - Regularity

$H = \{z > 0\}$



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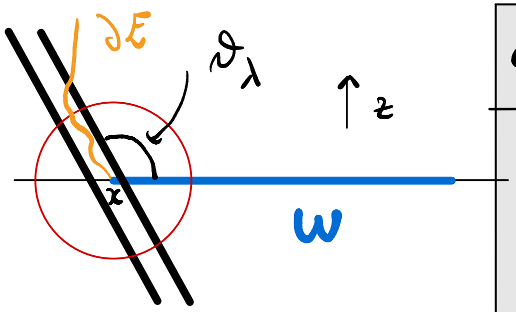

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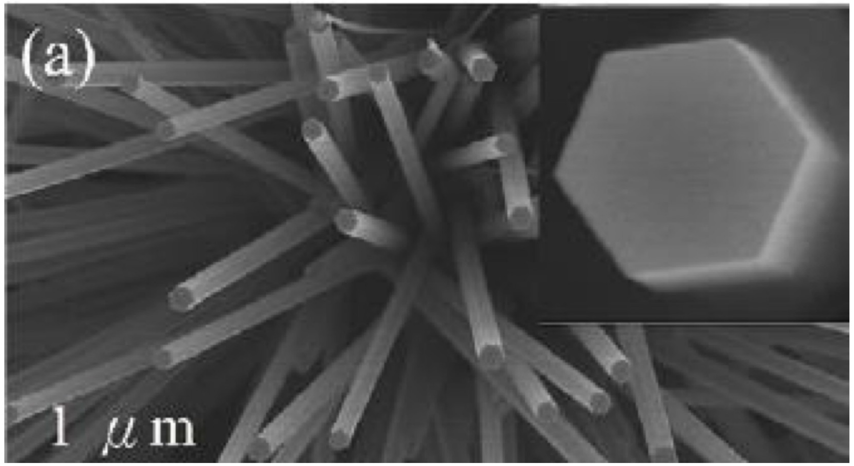
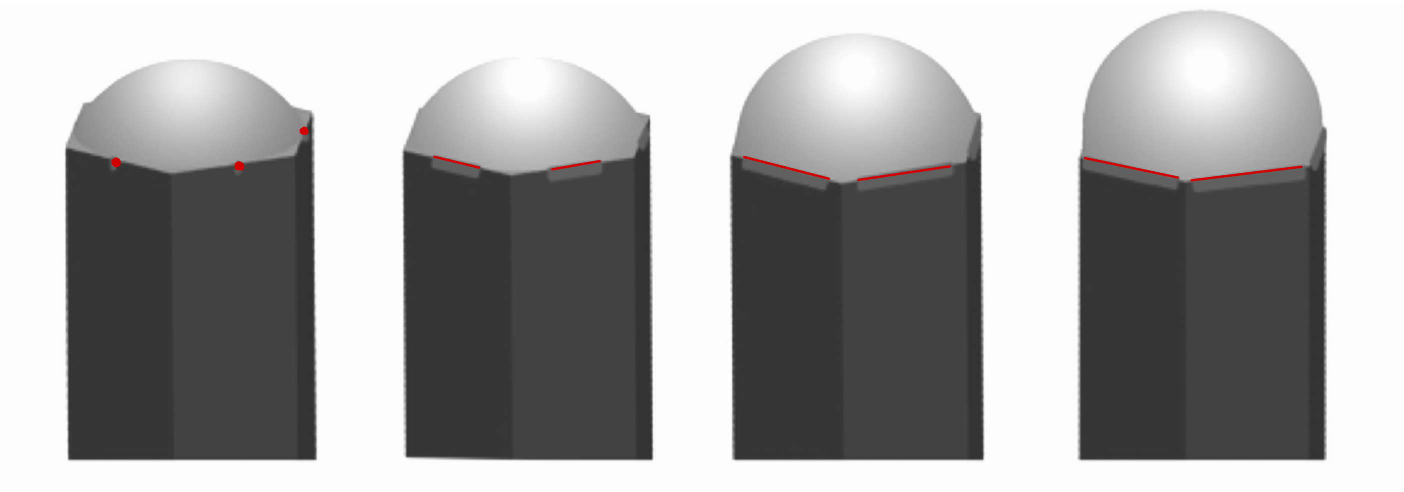
- ε - REGULARITY HOLDS in ANY DIMENSION
- extension to the BDRY of SAVIN'S PARTIAL HARNACK (2007), LINEARIZATION to SIGNORINI
  $\rightsquigarrow$ cfr FOCARDI - SPADARO for the NONPARAMETRIC CASE.

Nanowires with polygonal section





Krugstrop et al. PRL (2011)

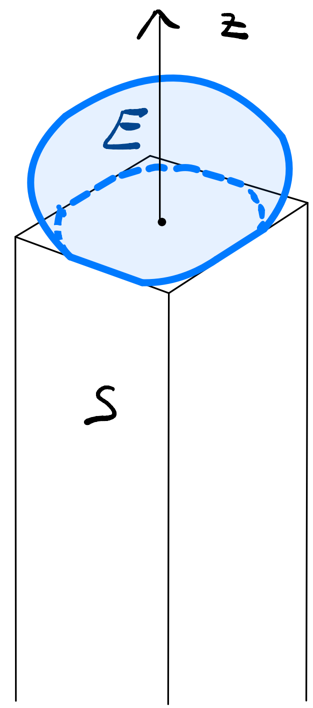
Nanowires with polygonal section – II

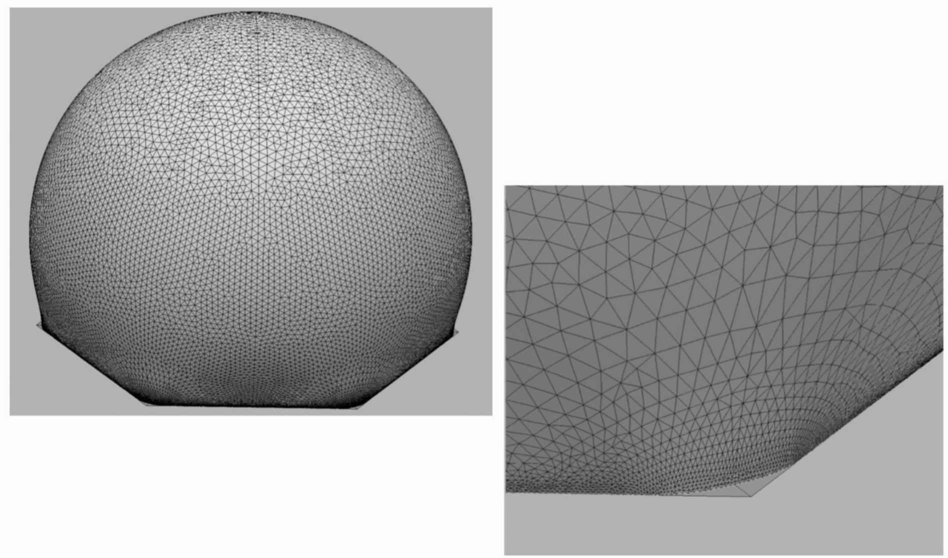
THEOREM (De Philippis – Fusco – M. '22)

- S with polygonal section ω
- $E \subseteq \{z > 0\}$ local minimizer of

$$\min\left\{\sigma(\partial^* F \setminus S) - \lambda\, \sigma(\partial^* F \cap \partial S) : F \subseteq H,\ \partial^* F \cap \partial H \subseteq \omega,\ |F| = m\right\},\ \lambda \in (0,1)$$

Then $\partial E \cap \{z > 0\}$ is a surface with boundary of class $C^{1,\alpha}$. Moreover, the contact line avoids the corners.





THANK you!