Lecture 6: Isoperimetric sets of large volume on spaces with nonnegative Ricci curvature and Euclidean volume growth

Elia Brué

Institute for Advanced Study, Princeton
elia.brue@ias.edu

June 23, 2022

Main result

Let (M^n, g) be a smooth Riemannian manifold satisfying:

- $\mathrm{Ric}_g \geq 0$;
- $\mathrm{AVR}(g) > 0$, where

$$\mathrm{AVR}(g) := \lim_{r \to \infty} \frac{\mathrm{Vol}_g(B_r(x))}{\omega_n r^n} .$$

Theorem (Antonelli-B.-Fogagnolo-Pozzetta)

If (M^n, g) satisfies a further assumption on the structure at infinity, then for any $V > \bar{V}$ there exists an isoperimetric region of volume V.

Corollary (Antonelli-B.-Fogagnolo-Pozzetta)

Let (M^n, g) be a manifold with $\mathrm{Sec}_g \geq 0$ and Euclidean volume growth. Then for any $V > \bar{V}$ there exists an isoperimetric region of volume V.

Main result

Let (M^n, g) be a smooth Riemannian manifold satisfying:

- $\mathrm{Ric}_g \geq 0$;
- $\mathrm{AVR}(g) > 0$, where

$$\mathrm{AVR}(g) := \lim_{r \to \infty} \frac{\mathrm{Vol}_g(B_r(x))}{\omega_n r^n} .$$

Theorem (Antonelli-B.-Fogagnolo-Pozzetta)

If (M^n, g) satisfies a further assumption on the structure at infinity, then for any $V > \bar{V}$ there exists an isoperimetric region of volume V.

Corollary (Antonelli-B.-Fogagnolo-Pozzetta)

Let (M^n, g) be a manifold with $\mathrm{Sec}_g \geq 0$ and Euclidean volume growth. Then for any $V > \bar{V}$ there exists an isoperimetric region of volume V.

Main result

Let (M^n, g) be a smooth Riemannian manifold satisfying:

- $\mathrm{Ric}_g \geq 0$;
- $\mathrm{AVR}(g) > 0$, where

$$\mathrm{AVR}(g) := \lim_{r \to \infty} \frac{\mathrm{Vol}_g(B_r(x))}{\omega_n r^n} .$$

Theorem (Antonelli-B.-Fogagnolo-Pozzetta)

If (M^n, g) satisfies a further assumption on the structure at infinity, then for any $V > \bar{V}$ there exists an isoperimetric region of volume V.

Corollary (Antonelli-B.-Fogagnolo-Pozzetta)

Let (M^n, g) be a manifold with $\mathrm{Sec}_g \geq 0$ and Euclidean volume growth. Then for any $V > \bar{V}$ there exists an isoperimetric region of volume V.

Plan of the talk

- Isoperimetric sets on non-smooth manifolds: lack of compactness and loss of mass
- The assumption on the structure at infinity
  - Motivation
  - Equivalent characterization
- Strategy of proof
- Open problems and related questions

Isoperimetric problem, non compact ambient space

- The family of sets with uniformly bounded mass and perimeter is not compact in L^1_{loc}.
- Enemy to existence: Minimizing sequences may lose mass at infinity.
- Key tool: Generalized existence of isoperimetric sets.
- Moral: Isoperimetric sets exist whenever escaping to infinity is not “isoperimetricaly convenient”.

Isoperimetric problem, non compact ambient space

- The family of sets with uniformly bounded mass and perimeter is not compact in L^1_{loc}.
- Enemy to existence: Minimizing sequences may lose mass at infinity.
- Key tool: Generalized existence of isoperimetric sets.
- Moral: Isoperimetric sets exist whenever escaping to infinity is not "isoperimetricaly convenient".

Isoperimetric problem, non compact ambient space

- The family of sets with uniformly bounded mass and perimeter is not compact in L^1_{loc}.
- Enemy to existence: Minimizing sequences may lose mass at infinity.
- Key tool: Generalized existence of isoperimetric sets.
- Moral: Isoperimetric sets exist whenever escaping to infinity is not “isoperimetricaly convenient”.

Isoperimetric problem, non compact ambient space

- The family of sets with uniformly bounded mass and perimeter is not compact in L^1_{loc}.
- Enemy to existence: Minimizing sequences may lose mass at infinity.
- Key tool: Generalized existence of isoperimetric sets.
- Moral: Isoperimetric sets exist whenever escaping to infinity is not "isoperimetricaly convenient".

Isoperimetric problem, non compact ambient space

- The family of sets with uniformly bounded mass and perimeter is not compact in L^1_{loc}.
- Enemy to existence: Minimizing sequences may lose mass at infinity.
- Key tool: Generalized existence of isoperimetric sets.
- Moral: Isoperimetric sets exist whenever escaping to infinity is not “isoperimetricaly convenient”.

Structure at infinity

- Tangent cones at infinity: Consider limit points of

$$\left(M, \frac{\mathsf{d}_g}{R_k}, \frac{\mathrm{Vol}_g}{R_k^n}, x\right) \xrightarrow{pmGH} (X, \mathsf{d}_X, \mathscr{H}_X^n, x)\,, \quad R_k \to \infty\,.$$

- Pointed limits at infinity: Let $x_k \to \infty$, consider limits

$$(M, \mathsf{d}_g, \mathrm{Vol}_g, x_k) \xrightarrow{pmGH} (Y, \rho, \mathscr{H}_Y^n, y)\,.$$

- Tangent cones to infinity and pointed limits are $\mathrm{RCD}(0, n)$ m.m.s. (and not more regular in general).

Structure at infinity

- Tangent cones at infinity: Consider limit points of

$$\left(M, \frac{\mathrm{d}_g}{R_k}, \frac{\mathrm{Vol}_g}{R_k^n}, x\right) \xrightarrow{pmGH} (X, \mathrm{d}_X, \mathscr{H}_X^n, x)\,, \quad R_k \to \infty\,.$$

- Pointed limits at infinity: Let $x_k \to \infty$, consider limits

$$(M, \mathrm{d}_g, \mathrm{Vol}_g, x_k) \xrightarrow{pmGH} (Y, \rho, \mathscr{H}_Y^n, y)\,.$$

- Tangent cones to infinity and pointed limits are $\mathrm{RCD}(0, n)$ m.m.s. (and not more regular in general).

Structure at infinity

- Tangent cones at infinity: Consider limit points of

$$\left(M, \frac{\mathsf{d}_g}{R_k}, \frac{\mathrm{Vol}_g}{R_k^n}, x\right) \xrightarrow{pmGH} (X, \mathsf{d}_X, \mathscr{H}_X^n, x), \quad R_k \to \infty .$$

- Pointed limits at infinity: Let $x_k \to \infty$, consider limits

$$(M, \mathsf{d}_g, \mathrm{Vol}_g, x_k) \xrightarrow{pmGH} (Y, \rho, \mathscr{H}_Y^n, y) .$$

- Tangent cones to infinity and pointed limits are $\mathsf{RCD}(0, n)$ m.m.s. (and not more regular in general).

Tangent cones are metric cones

Let $(X, \mathsf{d}_X, \mathscr{H}^n_X, x)$ be a tangent cone at infinity of (M^n, g).

- Volume rigidity:

$$\mathscr{H}^n_X(B_r(x)) = \mathrm{AVR}(g)\omega_n r^n\,, \quad \text{for every } r > 0\,.$$

- Volume cone implies metric cone: There exists $(Z, \mathsf{d}_Z, \mathscr{H}^{n-1}_Z)$, an $\mathrm{RCD}(n-1, n-2)$ m.m.s., such that

$$(X, \mathsf{d}_X, x) \simeq (C(Z), \mathsf{d}_C, O)\,, \quad O \text{ is a tip point}\,.$$

[Cheeger-Colding '96], [DePhilippis-Gigli '16].

Tangent cones are metric cones

Let $(X, \mathsf{d}_X, \mathscr{H}^n_X, x)$ be a tangent cone at infinity of (M^n, g).

- Volume rigidity:

$$\mathscr{H}^n_X(B_r(x)) = \mathrm{AVR}(g)\omega_n r^n\,, \quad \text{for every } r > 0\,.$$

- Volume cone implies metric cone: There exists $(Z, \mathsf{d}_Z, \mathscr{H}^{n-1}_Z)$, an $\mathrm{RCD}(n-1, n-2)$ m.m.s., such that

$$(X, \mathsf{d}_X, x) \simeq (C(Z), \mathsf{d}_C, O)\,, \quad O \text{ is a tip point}\,.$$

[Cheeger-Colding '96], [DePhilippis-Gigli '16].

Tangent cones are metric cones

Let $(X, \mathsf{d}_X, \mathscr{H}^n_X, x)$ be a tangent cone at infinity of (M^n, g).

- Volume rigidity:

$$\mathscr{H}^n_X(B_r(x)) = \mathrm{AVR}(g)\omega_n r^n\,, \quad \text{for every } r > 0\,.$$

- Volume cone implies metric cone: There exists $(Z, \mathsf{d}_Z, \mathscr{H}^{n-1}_Z)$, an $\mathsf{RCD}(n-1, n-2)$ m.m.s., such that

$$(X, \mathsf{d}_X, x) \simeq (C(Z), \mathsf{d}_C, O)\,, \quad O \text{ is a tip point}\,.$$

[Cheeger-Colding '96], [DePhilippis-Gigli '16].

Tangent cones are metric cones

Let $(X, \mathsf{d}_X, \mathscr{H}^n_X, x)$ be a tangent cone at infinity of (M^n, g).

- Volume rigidity:

$$\mathscr{H}^n_X(B_r(x)) = \mathrm{AVR}(g)\omega_n r^n\,, \quad \text{for every } r > 0\,.$$

- Volume cone implies metric cone: There exists $(Z, \mathsf{d}_Z, \mathscr{H}^{n-1}_Z)$, an $\mathsf{RCD}(n-1, n-2)$ m.m.s., such that

$$(X, \mathsf{d}_X, x) \simeq (C(Z), \mathsf{d}_C, O)\,, \quad O \text{ is a tip point}\,.$$

[Cheeger-Colding '96], [DePhilippis-Gigli '16].

The isoperimetric problem on cones

Theorem (Antonelli-Pasqualetto-Pozzetta-Semola)

Let $C(Z)$ be a metric cone over an $\mathrm{RCD}(n-1, n-2)$ m.m.s. $(Z, \mathsf{d}_Z, \mathscr{H}_Z^{n-1})$. Then, E is an isoperimetric set in $C(Z)$ iff it coincides with a ball centered at a tip point.

Proof.

- Let $O \in C(Z)$ a tip point, then $B_r(O)$ saturates the sharp isoperimetric inequality for any $r > 0$. Hence $E := B_r(O)$ is an isoperimetric set.
- If $E \subset C(Z)$ is isoperimetric, then it saturates the sharp isoperimetric inequality. We can apply the rigidity in the isoperimetric inequality in $\mathrm{RCD}(0, N)$ spaces [Antonelli-Pasqualetto-Pozzetta-Semola '22].

□

The isoperimetric problem on cones

Theorem (Antonelli-Pasqualetto-Pozzetta-Semola)

Let $C(Z)$ be a metric cone over an $\mathrm{RCD}(n-1, n-2)$ *m.m.s.* $(Z, \mathsf{d}_Z, \mathscr{H}^{n-1}_Z)$. *Then, E is an isoperimetric set in $C(Z)$ iff it coincides with a ball centered at a tip point.*

Proof.

- Let $O \in C(Z)$ a tip point, then $B_r(O)$ saturates the sharp isoperimetric inequality for any $r > 0$. Hence $E := B_r(O)$ is an isoperimetric set.
- If $E \subset C(Z)$ is isoperimetric, then it saturates the sharp isoperimetric inequality. We can apply the rigidity in the isoperimetric inequality in $\mathrm{RCD}(0, N)$ spaces [Antonelli-Pasqualetto-Pozzetta-Semola '22].

□

The isoperimetric problem on cones

Theorem (Antonelli-Pasqualetto-Pozzetta-Semola)

Let $C(Z)$ be a metric cone over an $\mathrm{RCD}(n-1, n-2)$ m.m.s. $(Z, \mathsf{d}_Z, \mathscr{H}_Z^{n-1})$. Then, E is an isoperimetric set in $C(Z)$ iff it coincides with a ball centered at a tip point.

Proof.

- Let $O \in C(Z)$ a tip point, then $B_r(O)$ saturates the sharp isoperimetric inequality for any $r > 0$. Hence $E := B_r(O)$ is an isoperimetric set.
- If $E \subset C(Z)$ is isoperimetric, then it saturates the sharp isoperimetric inequality. We can apply the rigidity in the isoperimetric inequality in $\mathrm{RCD}(0, N)$ spaces [Antonelli-Pasqualetto-Pozzetta-Semola '22].

□

The isoperimetric problem on cones

Theorem (Antonelli-Pasqualetto-Pozzetta-Semola)

Let $C(Z)$ be a metric cone over an $\mathrm{RCD}(n-1, n-2)$ *m.m.s.* $(Z, \mathsf{d}_Z, \mathscr{H}_Z^{n-1})$*. Then, E is an isoperimetric set in $C(Z)$ iff it coincides with a ball centered at a tip point.*

Proof.

- Let $O \in C(Z)$ a tip point, then $B_r(O)$ saturates the sharp isoperimetric inequality for any $r > 0$. Hence $E := B_r(O)$ is an isoperimetric set.

- If $E \subset C(Z)$ is isoperimetric, then it saturates the sharp isoperimetric inequality. We can apply the rigidity in the isoperimetric inequality in $\mathrm{RCD}(0, N)$ spaces [Antonelli-Pasqualetto-Pozzetta-Semola '22].

$\square$

The isoperimetric problem on cones

- Uniqueness: Isoperimetric sets are unique iff there exists a unique tip point $O \in C(Z)$.

  Uniqueness of the tip point iff $C(Z)$ does not split any line.

- ε-Stability: If $C(Z)$ does not split any line, then isoperimetric sets are stable.
  - Second variation: Pick $B_1(O)$, consider the perturbation

    $$u \in C_0(Z) \to \Sigma_u := \{(u(z)+1, z) \,:\, z \in Z\} \subset C(Z)\,.$$

    The second variation of $u \to \mathrm{Per}(\Sigma_u)$ gives the Jacobi operator

    $$u \to \mathcal{L}_Z u := -\Delta_Z u - (n-1)u\,.$$

  - Rigidity in Obata's theorem: If $C(Z)$ does not split, then $\mathcal{L}_Z \geq \varepsilon$ for some $\varepsilon > 0$.

The isoperimetric problem on cones

- Uniqueness: Isoperimetric sets are unique iff there exists a unique tip point $O \in C(Z)$.

  Uniqueness of the tip point iff $C(Z)$ does not split any line.

- ε-Stability: If $C(Z)$ does not split any line, then isoperimetric sets are stable.
  - Second variation: Pick $B_1(O)$, consider the perturbation

    $$u \in C_0(Z) \to \Sigma_u := \{(u(z)+1, z) \,:\, z \in Z\} \subset C(Z)\,.$$

    The second variation of $u \to \mathrm{Per}(\Sigma_u)$ gives the Jacobi operator

    $$u \to \mathcal{L}_Z u := -\Delta_Z u - (n-1)u\,.$$

  - Rigidity in Obata's theorem: If $C(Z)$ does not split, then $\mathcal{L}_Z \geq \varepsilon$ for some $\varepsilon > 0$.

The isoperimetric problem on cones

- Uniqueness: Isoperimetric sets are unique iff there exists a unique tip point $O \in C(Z)$.

  Uniqueness of the tip point iff $C(Z)$ does not split any line.

- ε-Stability: If $C(Z)$ does not split any line, then isoperimetric sets are stable.
  - Second variation: Pick $B_1(O)$, consider the perturbation

    $$u \in C_0(Z) \to \Sigma_u := \{(u(z)+1, z) \, : \, z \in Z\} \subset C(Z) \, .$$

    The second variation of $u \to \mathrm{Per}(\Sigma_u)$ gives the Jacobi operator

    $$u \to \mathcal{L}_Z u := -\Delta_Z u - (n-1)u \, .$$

  - Rigidity in Obata's theorem: If $C(Z)$ does not split, then $\mathcal{L}_Z \geq \varepsilon$ for some $\varepsilon > 0$.

The isoperimetric problem on cones

- Uniqueness: Isoperimetric sets are unique iff there exists a unique tip point $O \in C(Z)$.

  Uniqueness of the tip point iff $C(Z)$ does not split any line.

- ε-Stability: If $C(Z)$ does not split any line, then isoperimetric sets are stable.
  - Second variation: Pick $B_1(O)$, consider the perturbation

    $$u \in C_0(Z) \to \Sigma_u := \{(u(z)+1, z) \, : \, z \in Z\} \subset C(Z)\,.$$

    The second variation of $u \to \mathrm{Per}(\Sigma_u)$ gives the Jacobi operator

    $$u \to \mathcal{L}_Z u := -\Delta_Z u - (n-1)u\,.$$

  - Rigidity in Obata's theorem: If $C(Z)$ does not split, then $\mathcal{L}_Z \geq \varepsilon$ for some $\varepsilon > 0$.

The isoperimetric problem on cones

- Uniqueness: Isoperimetric sets are unique iff there exists a unique tip point $O \in C(Z)$.

  Uniqueness of the tip point iff $C(Z)$ does not split any line.

- ε-Stability: If $C(Z)$ does not split any line, then isoperimetric sets are stable.
  - Second variation: Pick $B_1(O)$, consider the perturbation

    $$u \in C_0(Z) \to \Sigma_u := \{(u(z)+1, z) \, : \, z \in Z\} \subset C(Z) \, .$$

    The second variation of $u \to \mathrm{Per}(\Sigma_u)$ gives the Jacobi operator

    $$u \to \mathcal{L}_Z u := -\Delta_Z u - (n-1)u \, .$$

  - Rigidity in Obata's theorem: If $C(Z)$ does not split, then $\mathcal{L}_Z \geq \varepsilon$ for some $\varepsilon > 0$.

The isoperimetric problem on cones

- Uniqueness: Isoperimetric sets are unique iff there exists a unique tip point $O \in C(Z)$.

  Uniqueness of the tip point iff $C(Z)$ does not split any line.

- ε-Stability: If $C(Z)$ does not split any line, then isoperimetric sets are stable.
  - Second variation: Pick $B_1(O)$, consider the perturbation

    $$u \in C_0(Z) \to \Sigma_u := \{(u(z)+1, z) \, : \, z \in Z\} \subset C(Z)\,.$$

    The second variation of $u \to \mathrm{Per}(\Sigma_u)$ gives the Jacobi operator

    $$u \to \mathcal{L}_Z u := -\Delta_Z u - (n-1)u\,.$$

  - Rigidity in Obata's theorem: If $C(Z)$ does not split, then $\mathcal{L}_Z \geq \varepsilon$ for some $\varepsilon > 0$.

What we have seen so far

Let (M^n, g) satisfy $\mathrm{Ric}_g \geq 0$ and $\mathrm{AVR}(g) > 0$.

- Tangent cone at infinity: At big scales, $(M, \mathrm{d}_g, \mathrm{Vol}_g)$ is GH-close to an $\mathrm{RCD}(0, n)$ cone;
- Isoperimetric problem on cones: Tangent cones at infinity to M admit a unique and ε-stable isoperimetric set for each volume $V > 0$, provided they do not split a line.

Theorem (Perturbation, rough)

Fix $\varepsilon > 0$. If at any sufficiently big scale (M, g) is close to a model space admitting ε-stable isoperimetric sets of each volume, then (M, g) admits isoperimetric regions of big volume.

Assumption on the structure at infinity

No tangent cone at infinity splits a line.

What we have seen so far

Let (M^n, g) satisfy $\mathrm{Ric}_g \geq 0$ and $\mathrm{AVR}(g) > 0$.

- Tangent cone at infinity: At big scales, $(M, \mathsf{d}_g, \mathrm{Vol}_g)$ is GH-close to an $\mathrm{RCD}(0, n)$ cone;
- Isoperimetric problem on cones: Tangent cones at infinity to M admit a unique and ε-stable isoperimetric set for each volume $V > 0$, provided they do not split a line.

Theorem (Perturbation, rough)

Fix $\varepsilon > 0$. If at any sufficiently big scale (M, g) is close to a model space admitting ε-stable isoperimetric sets of each volume, then (M, g) admits isoperimetric regions of big volume.

Assumption on the structure at infinity

No tangent cone at infinity splits a line.

What we have seen so far

Let (M^n, g) satisfy $\mathrm{Ric}_g \geq 0$ and $\mathrm{AVR}(g) > 0$.

- Tangent cone at infinity: At big scales, $(M, \mathsf{d}_g, \mathrm{Vol}_g)$ is GH-close to an $\mathrm{RCD}(0, n)$ cone;
- Isoperimetric problem on cones: Tangent cones at infinity to M admit a unique and ε-stable isoperimetric set for each volume $V > 0$, provided they do not split a line.

Theorem (Perturbation, rough)

Fix $\varepsilon > 0$. If at any sufficiently big scale (M, g) is close to a model space admitting ε-stable isoperimetric sets of each volume, then (M, g) admits isoperimetric regions of big volume.

Assumption on the structure at infinity

No tangent cone at infinity splits a line.

What we have seen so far

Let (M^n, g) satisfy $\mathrm{Ric}_g \geq 0$ and $\mathrm{AVR}(g) > 0$.

- Tangent cone at infinity: At big scales, $(M, \mathsf{d}_g, \mathrm{Vol}_g)$ is GH-close to an $\mathrm{RCD}(0, n)$ cone;
- Isoperimetric problem on cones: Tangent cones at infinity to M admit a unique and ε-stable isoperimetric set for each volume $V > 0$, provided they do not split a line.

Theorem (Perturbation, rough)

Fix $\varepsilon > 0$. If at any sufficiently big scale (M, g) is close to a model space admitting ε-stable isoperimetric sets of each volume, then (M, g) admits isoperimetric regions of big volume.

Assumption on the structure at infinity

No tangent cone at infinity splits a line.

What we have seen so far

Let (M^n, g) satisfy $\mathrm{Ric}_g \geq 0$ and $\mathrm{AVR}(g) > 0$.

- Tangent cone at infinity: At big scales, $(M, \mathsf{d}_g, \mathrm{Vol}_g)$ is GH-close to an $\mathrm{RCD}(0, n)$ cone;
- Isoperimetric problem on cones: Tangent cones at infinity to M admit a unique and ε-stable isoperimetric set for each volume $V > 0$, provided they do not split a line.

Theorem (Perturbation, rough)

Fix $\varepsilon > 0$. If at any sufficiently big scale (M, g) is close to a model space admitting ε-stable isoperimetric sets of each volume, then (M, g) admits isoperimetric regions of big volume.

Assumption on the structure at infinity

No tangent cone at infinity splits a line.

What we have seen so far

Let (M^n, g) satisfy $\mathrm{Ric}_g \geq 0$ and $\mathrm{AVR}(g) > 0$.

- Tangent cone at infinity: At big scales, $(M, \mathsf{d}_g, \mathrm{Vol}_g)$ is GH-close to an $\mathrm{RCD}(0, n)$ cone;
- Isoperimetric problem on cones: Tangent cones at infinity to M admit a unique and ε-stable isoperimetric set for each volume $V > 0$, provided they do not split a line.

Theorem (Perturbation, rough)

Fix $\varepsilon > 0$. If at any sufficiently big scale (M, g) is close to a model space admitting ε-stable isoperimetric sets of each volume, then (M, g) admits isoperimetric regions of big volume.

Assumption on the structure at infinity

No tangent cone at infinity splits a line.

Assumption on the structure at infinity

Write $M^n = N^{n-k} \times \mathbb{R}^k$, for some $k \leq n$, where N does not split any line.

> **Assumption on the structure at infinity**
> No tangent cone at infinity to N splits a line.

Topogonov's theorem implies the following.

Lemma
If $M^n = N^{n-k} \times \mathbb{R}^k$ satisfies $\mathrm{Sec}_g \geq 0$ and $\mathrm{AVR}(g) > 0$, then the tangent cone at infinity to N is unique and does not split any line.

Assumption on the structure at infinity

Write $M^n = N^{n-k} \times \mathbb{R}^k$, for some $k \leq n$, where N does not split any line.

Assumption on the structure at infinity

No tangent cone at infinity to N splits a line.

Topogonov's theorem implies the following.

Lemma

If $M^n = N^{n-k} \times \mathbb{R}^k$ satisfies $\mathrm{Sec}_g \geq 0$ and $\mathrm{AVR}(g) > 0$, then the tangent cone at infinity to N is unique and does not split any line.

Assumption on the structure at infinity

Write $M^n = N^{n-k} \times \mathbb{R}^k$, for some $k \leq n$, where N does not split any line.

Assumption on the structure at infinity

No tangent cone at infinity to N splits a line.

Topogonov's theorem implies the following.

Lemma

If $M^n = N^{n-k} \times \mathbb{R}^k$ satisfies $\mathrm{Sec}_g \geq 0$ and $\mathrm{AVR}(g) > 0$, then the tangent cone at infinity to N is unique and does not split any line.

Assumption on the structure at infinity

Write $M^n = N^{n-k} \times \mathbb{R}^k$, for some $k \leq n$, where N does not split any line.

Assumption on the structure at infinity

No tangent cone at infinity to N splits a line.

Topogonov's theorem implies the following.

Lemma

If $M^n = N^{n-k} \times \mathbb{R}^k$ satisfies $\mathrm{Sec}_g \geq 0$ and $\mathrm{AVR}(g) > 0$, then the tangent cone at infinity to N is unique and does not split any line.

Main result

Theorem (Antonelli-B.-Fogagnolo-Pozzetta)

Let (M^n, g) be a manifold with Ric ≥ 0 and Euclidean volume growth, i.e.

$$\lim_{r\to\infty} \frac{Vol_g(B_r(x))}{\omega_n r^n} = \mathrm{AVR}(g) \in (0,1)\,.$$

Write $M^n = N^{n-k} \times \mathbb{R}^k$, $0 \leq k < n$ where N^{n-k} does not split any line, and assume that no tangent cone at infinity to N^{n-k} splits a line. Then for any $V > \bar{V}$ there exists an isoperimetric region of volume V.

"If all the models at infinity of N have ε-stable isoperimetric sets, then (M, g) admits isoperimetric sets of big volume".

Main result

Theorem (Antonelli-B.-Fogagnolo-Pozzetta)

Let (M^n, g) be a manifold with Ric ≥ 0 and Euclidean volume growth, i.e.

$$\lim_{r\to\infty} \frac{Vol_g(B_r(x))}{\omega_n r^n} = \mathrm{AVR}(g) \in (0,1)\,.$$

Write $M^n = N^{n-k} \times \mathbb{R}^k$, $0 \leq k < n$ where N^{n-k} does not split any line, and assume that no tangent cone at infinity to N^{n-k} splits a line. Then for any $V > \bar{V}$ there exists an isoperimetric region of volume V.

“If all the models at infinity of N have ε-stable isoperimetric sets, then (M, g) admits isoperimetric sets of big volume”.

Characterization in terms of pointed limits at infinity

Definition (Pointed limits to infinity)

$$\mathcal{F}_\infty(M,g) := \{(Y,\rho,\mathscr{H}^n_Y,y) \,:\, \text{“}x_k \to y\text{''}\}\,,$$

i.e. $\mathcal{F}_\infty(M,g)$ is the collection of pmGH-limits

$$(M,\mathrm{d}_g,\mathrm{Vol}_g,x_k) \xrightarrow{pmGH} (Y,\rho,\mathscr{H}^n_Y,y)\,, \quad \text{where } x_k \to \infty\,.$$

Characterization in terms of pointed limits at infinity

Definition (Pointed limits to infinity)

$$\mathcal{F}_\infty(M,g) := \{(Y,\rho,\mathscr{H}^n_Y,y)\,:\, \text{“}x_k \to y\text{”}\}\,,$$

i.e. $\mathcal{F}_\infty(M,g)$ is the collection of pmGH-limits

$$(M,\mathrm{d}_g,\mathrm{Vol}_g,x_k) \xrightarrow{pmGH} (Y,\rho,\mathscr{H}^n_Y,y)\,, \quad \text{where } x_k \to \infty\,.$$

Assumption II

Assume for simplicity that M does not split any line.

Assumption II

There exists $\varepsilon > 0$ such that

$$\mathrm{AVR}(Y) \geq \mathrm{AVR}(g) + \varepsilon \qquad \forall\, (Y, \rho, \mathscr{H}^n_Y) \in \mathcal{F}_\infty(M, g)\,.$$

$$\mathrm{AVR}(Y) := \lim_{r \to \infty} \frac{\mathscr{H}^n_Y(B_r(y))}{\omega_n r^n}\,.$$

Proposition

If no tangent cone at infinity of M splits a line, then Assumption II is satisfied.

Assumption II

Assume for simplicity that M does not split any line.

Assumption II

There exists $\varepsilon > 0$ such that

$$\mathrm{AVR}(Y) \geq \mathrm{AVR}(g) + \varepsilon \qquad \forall\, (Y, \rho, \mathscr{H}^n_Y) \in \mathcal{F}_\infty(M, g)\,.$$

$$\mathrm{AVR}(Y) := \lim_{r\to\infty} \frac{\mathscr{H}^n_Y(B_r(y))}{\omega_n r^n}\,.$$

Proposition

If no tangent cone at infinity of M splits a line, then Assumption II is satisfied.

Assumption II

Assume for simplicity that M does not split any line.

Assumption II

There exists $\varepsilon > 0$ such that

$$\mathrm{AVR}(Y) \geq \mathrm{AVR}(g) + \varepsilon \qquad \forall\, (Y, \rho, \mathscr{H}^n_Y) \in \mathcal{F}_\infty(M, g)\,.$$

$$\mathrm{AVR}(Y) := \lim_{r \to \infty} \frac{\mathscr{H}^n_Y(B_r(y))}{\omega_n r^n}\,.$$

Proposition

If no tangent cone at infinity of M splits a line, then Assumption II is satisfied.

Assumption II

Assume for simplicity that M does not split any line.

Assumption II

There exists $\varepsilon > 0$ such that

$$\mathrm{AVR}(Y) \geq \mathrm{AVR}(g) + \varepsilon \qquad \forall\, (Y, \rho, \mathscr{H}^n_Y) \in \mathcal{F}_\infty(M, g)\,.$$

$$\mathrm{AVR}(Y) := \lim_{r \to \infty} \frac{\mathscr{H}^n_Y(B_r(y))}{\omega_n r^n}\,.$$

Proposition

If no tangent cone at infinity of M splits a line, then Assumption II is satisfied.

Assumption II

Assume for simplicity that M does not split any line.

Assumption II

There exists $\varepsilon > 0$ such that

$$\mathrm{AVR}(Y) \geq \mathrm{AVR}(g) + \varepsilon \qquad \forall\, (Y, \rho, \mathscr{H}^n_Y) \in \mathcal{F}_\infty(M, g)\,.$$

$$\mathrm{AVR}(Y) := \lim_{r\to\infty} \frac{\mathscr{H}^n_Y(B_r(y))}{\omega_n r^n}\,.$$

Proposition

If no tangent cone at infinity of M splits a line, then Assumption II is satisfied.

- Let $y_k \in M$, $y_k \to \infty$ such that

$$(M, \mathsf{d}_g, \mathrm{Vol}_g, y_k) \xrightarrow{pmGH} (Y, \rho, \mathscr{H}^n_Y, y)\,, \quad k \to \infty\,.$$

Fix $x \in M$, set $R_k := \mathsf{d}(y_k, x)$, up to subsequence

$$\left(M, \frac{\mathsf{d}_g}{R_k}, \frac{\mathrm{Vol}_g}{R_k^n}, x\right) \xrightarrow{pmGH} (C(Z), \mathsf{d}_{C(Z)}, \mathscr{H}^n_{C(Z)}, O)\,, \quad y_k \to y_\infty \in C(Z)\,.$$

Notice that $\mathsf{d}_{C(Z)}(y_\infty, O) = 1$, hence y_∞ is not a tip point.

- Volume monotonicity:

$$\lim_{r \to 0} \frac{\mathscr{H}^n_{C(Z)}(B_r(y_\infty))}{\omega_n r^n} \le \mathrm{AVR}(Y)\,.$$

- Cone splitting, Gromov's compactness theorem: there exists $\varepsilon > 0$ such that, for any tangent cone at infinity $(C(Z), O)$ it holds

$$\lim_{r \to 0} \frac{\mathscr{H}^n_{C(X)}(B_r(y_\infty))}{\omega_n r^n} > \lim_{r \to 0} \frac{\mathscr{H}^n_{C(X)}(B_r(O))}{\omega_n r^n} + \varepsilon = \mathrm{AVR}(g) + \varepsilon$$

since y_∞ is not a tip point.

- Let $y_k \in M$, $y_k \to \infty$ such that

$$(M, \mathsf{d}_g, \mathrm{Vol}_g, y_k) \xrightarrow{pmGH} (Y, \rho, \mathscr{H}^n_Y, y)\,, \quad k \to \infty\,.$$

Fix $x \in M$, set $R_k := \mathsf{d}(y_k, x)$, up to subsequence

$$\left(M, \frac{\mathsf{d}_g}{R_k}, \frac{\mathrm{Vol}_g}{R_k^n}, x\right) \xrightarrow{pmGH} (C(Z), \mathsf{d}_{C(Z)}, \mathscr{H}^n_{C(Z)}, O)\,, \quad y_k \to y_\infty \in C(Z)\,.$$

Notice that $\mathsf{d}_{C(Z)}(y_\infty, O) = 1$, hence y_∞ is not a tip point.

- Volume monotonicity:

$$\lim_{r \to 0} \frac{\mathscr{H}^n_{C(Z)}(B_r(y_\infty))}{\omega_n r^n} \leq \mathrm{AVR}(Y)\,.$$

- Cone splitting, Gromov's compactness theorem: there exists $\varepsilon > 0$ such that, for any tangent cone at infinity $(C(Z), O)$ it holds

$$\lim_{r \to 0} \frac{\mathscr{H}^n_{C(X)}(B_r(y_\infty))}{\omega_n r^n} > \lim_{r \to 0} \frac{\mathscr{H}^n_{C(X)}(B_r(O))}{\omega_n r^n} + \varepsilon = \mathrm{AVR}(g) + \varepsilon$$

since y_∞ is not a tip point.

- Let $y_k \in M$, $y_k \to \infty$ such that

$$(M, \mathsf{d}_g, \mathrm{Vol}_g, y_k) \xrightarrow{pmGH} (Y, \rho, \mathscr{H}^n_Y, y), \quad k \to \infty .$$

Fix $x \in M$, set $R_k := \mathsf{d}(y_k, x)$, up to subsequence

$$\left(M, \frac{\mathsf{d}_g}{R_k}, \frac{\mathrm{Vol}_g}{R_k^n}, x\right) \xrightarrow{pmGH} (C(Z), \mathsf{d}_{C(Z)}, \mathscr{H}^n_{C(Z)}, O), \quad y_k \to y_\infty \in C(Z) .$$

Notice that $\mathsf{d}_{C(Z)}(y_\infty, O) = 1$, hence y_∞ is not a tip point.

- Volume monotonicity:

$$\lim_{r \to 0} \frac{\mathscr{H}^n_{C(Z)}(B_r(y_\infty))}{\omega_n r^n} \leq \mathrm{AVR}(Y) .$$

- Cone splitting, Gromov's compactness theorem: there exists $\varepsilon > 0$ such that, for any tangent cone at infinity $(C(Z), O)$ it holds

$$\lim_{r \to 0} \frac{\mathscr{H}^n_{C(X)}(B_r(y_\infty))}{\omega_n r^n} > \lim_{r \to 0} \frac{\mathscr{H}^n_{C(X)}(B_r(O))}{\omega_n r^n} + \varepsilon = \mathrm{AVR}(g) + \varepsilon$$

since y_∞ is not a tip point.

- Let $y_k \in M$, $y_k \to \infty$ such that

$$(M, \mathrm{d}_g, \mathrm{Vol}_g, y_k) \xrightarrow{pmGH} (Y, \rho, \mathscr{H}^n_Y, y)\,, \quad k \to \infty\,.$$

Fix $x \in M$, set $R_k := \mathrm{d}(y_k, x)$, up to subsequence

$$\left(M, \frac{\mathrm{d}_g}{R_k}, \frac{\mathrm{Vol}_g}{R_k^n}, x\right) \xrightarrow{pmGH} (C(Z), \mathrm{d}_{C(Z)}, \mathscr{H}^n_{C(Z)}, O)\,, \quad y_k \to y_\infty \in C(Z)\,.$$

Notice that $\mathrm{d}_{C(Z)}(y_\infty, O) = 1$, hence y_∞ is not a tip point.

- Volume monotonicity:

$$\lim_{r\to 0} \frac{\mathscr{H}^n_{C(Z)}(B_r(y_\infty))}{\omega_n r^n} \leq \mathrm{AVR}(Y)\,.$$

- Cone splitting, Gromov's compactness theorem: there exists $\varepsilon > 0$ such that, for any tangent cone at infinity $(C(Z), O)$ it holds

$$\lim_{r\to 0} \frac{\mathscr{H}^n_{C(X)}(B_r(y_\infty))}{\omega_n r^n} > \lim_{r\to 0} \frac{\mathscr{H}^n_{C(X)}(B_r(O))}{\omega_n r^n} + \varepsilon = \mathrm{AVR}(g) + \varepsilon$$

since y_∞ is not a tip point.

- Let $y_k \in M$, $y_k \to \infty$ such that

$$(M, \mathsf{d}_g, \mathrm{Vol}_g, y_k) \xrightarrow{pmGH} (Y, \rho, \mathscr{H}^n_Y, y)\,, \quad k \to \infty\,.$$

Fix $x \in M$, set $R_k := \mathsf{d}(y_k, x)$, up to subsequence

$$\left(M, \frac{\mathsf{d}_g}{R_k}, \frac{\mathrm{Vol}_g}{R_k^n}, x\right) \xrightarrow{pmGH} (C(Z), \mathsf{d}_{C(Z)}, \mathscr{H}^n_{C(Z)}, O)\,, \quad y_k \to y_\infty \in C(Z)\,.$$

Notice that $\mathsf{d}_{C(Z)}(y_\infty, O) = 1$, hence y_∞ is not a tip point.

- Volume monotonicity:

$$\lim_{r \to 0} \frac{\mathscr{H}^n_{C(Z)}(B_r(y_\infty))}{\omega_n r^n} \leq \mathrm{AVR}(Y)\,.$$

- Cone splitting, Gromov's compactness theorem: there exists $\varepsilon > 0$ such that, for any tangent cone at infinity $(C(Z), O)$ it holds

$$\lim_{r \to 0} \frac{\mathscr{H}^n_{C(X)}(B_r(y_\infty))}{\omega_n r^n} > \lim_{r \to 0} \frac{\mathscr{H}^n_{C(X)}(B_r(O))}{\omega_n r^n} + \varepsilon = \mathrm{AVR}(g) + \varepsilon$$

since y_∞ is not a tip point.

Ingredients of proof

- Generalized existence theorem
- Concavity of the isoperimetric profile
- Sharp Isoperimetric inequality on $\mathsf{RCD}(0, N)$ spaces

Strategy of proof

Fix $V > 0$. Let $E_k \subset M$ be a minimizing sequence of volume V.

Theorem (Generalized existence)

There exist $(Y_1, \rho_1, y_1), \ldots, (Y_m, \rho_m, y_m)$*, pointed limits at infinity, and isoperimetric sets*

$$E^0 \subset M\,, \quad E^1 \subset Y_1\,, \ldots, E^m \subset Y_m\,,$$

such that

$$E_k \to E^0 \cup E^1 \cup \ldots \cup E^m \quad \text{in } L^1 \text{ as } k \to \infty\,.$$

Moreover

$$|E^0| + |E^1| + \ldots + |E^m| = V\,,$$

and

$$\mathrm{Per}(E^0) + \mathrm{Per}(E^1) + \ldots + \mathrm{Per}(E^m) \leq I_M(V)\,.$$

Strategy of proof

Fix $V > 0$. Let $E_k \subset M$ be a minimizing sequence of volume V.

Theorem (Generalized existence)

There exist $(Y_1, \rho_1, y_1), \ldots, (Y_m, \rho_m, y_m)$*, pointed limits at infinity, and isoperimetric sets*

$$E^0 \subset M\,, \quad E^1 \subset Y_1\,, \ldots, E^m \subset Y_m\,,$$

such that

$$E_k \to E^0 \cup E^1 \cup \ldots \cup E^m \quad \text{in } L^1 \text{ as } k \to \infty\,.$$

Moreover

$$|E^0| + |E^1| + \ldots + |E^m| = V\,,$$

and

$$\mathrm{Per}(E^0) + \mathrm{Per}(E^1) + \ldots + \mathrm{Per}(E^m) \le I_M(V)\,.$$

Strategy of proof

Fix $V > 0$. Let $E_k \subset M$ be a minimizing sequence of volume V.

Theorem (Generalized existence)

There exist $(Y_1, \rho_1, y_1), \ldots, (Y_m, \rho_m, y_m)$, *pointed limits at infinity, and isoperimetric sets*

$$E^0 \subset M, \quad E^1 \subset Y_1, \ldots, E^m \subset Y_m,$$

such that

$$E_k \to E^0 \cup E^1 \cup \ldots \cup E^m \quad \text{in } L^1 \text{ as } k \to \infty.$$

Moreover

$$|E^0| + |E^1| + \ldots + |E^m| = V,$$

and

$$\mathrm{Per}(E^0) + \mathrm{Per}(E^1) + \ldots + \mathrm{Per}(E^m) \le I_M(V).$$

Strategy of proof

Fix $V > 0$. Let $E_k \subset M$ be a minimizing sequence of volume V.

Theorem (Generalized existence)

There exist $(Y_1, \rho_1, y_1), \ldots, (Y_m, \rho_m, y_m)$*, pointed limits at infinity, and isoperimetric sets*

$$E^0 \subset M\,, \quad E^1 \subset Y_1\,, \ldots, E^m \subset Y_m\,,$$

such that

$$E_k \to E^0 \cup E^1 \cup \ldots \cup E^m \quad \text{in } L^1 \text{ as } k \to \infty\,.$$

Moreover

$$|E^0| + |E^1| + \ldots + |E^m| = V\,,$$

and

$$\mathrm{Per}(E^0) + \mathrm{Per}(E^1) + \ldots + \mathrm{Per}(E^m) \leq I_M(V)\,.$$

Concavity of the isoperimetric profile

Theorem (Antonelli-B.-Fogagnolo-Pozzetta)

If (M^n, g) is non-collapsed and $Ric \geq 0$ then $I_M^{\frac{n}{n-1}}$ is concave. In particular I_M is strictly subadditive.

$$I_M(V) \leq I_M(|E^0|) + \ldots + I_M(|E^m|) \leq \mathrm{Per}(E^0) + \ldots + \mathrm{Per}(E^m) \leq I_M(V)$$

Corollary

There exists $0 \leq i \leq m$ such that $E^j = \emptyset$ for $j \neq i$.

[Bavard, Pansu '86], [Bayle '03], [Mondino, Nardulli '16], [Antonelli, B., Fogagnolo, Pozzetta '21].

Concavity of the isoperimetric profile

Theorem (Antonelli-B.-Fogagnolo-Pozzetta)

If (M^n, g) is non-collapsed and $Ric \geq 0$ then $I_M^{\frac{n}{n-1}}$ is concave. In particular I_M is strictly subadditive.

$$I_M(V) \leq I_M(|E^0|) + \ldots + I_M(|E^m|) \leq \mathrm{Per}(E^0) + \ldots + \mathrm{Per}(E^m) \leq I_M(V)$$

Corollary

There exists $0 \leq i \leq m$ such that $E^j = \emptyset$ for $j \neq i$.

[Bavard, Pansu '86], [Bayle '03], [Mondino, Nardulli '16], [Antonelli, B., Fogagnolo, Pozzetta '21].

Concavity of the isoperimetric profile

Theorem (Antonelli-B.-Fogagnolo-Pozzetta)

If (M^n, g) is non-collapsed and $Ric \geq 0$ then $I_M^{\frac{n}{n-1}}$ is concave. In particular I_M is strictly subadditive.

$$I_M(V) \leq I_M(|E^0|) + \ldots + I_M(|E^m|) \leq \mathrm{Per}(E^0) + \ldots + \mathrm{Per}(E^m) \leq I_M(V)$$

Corollary

There exists $0 \leq i \leq m$ such that $E^j = \emptyset$ for $j \neq i$.

[Bavard, Pansu '86], [Bayle '03], [Mondino, Nardulli '16], [Antonelli, B., Fogagnolo, Pozzetta '21].

Concavity of the isoperimetric profile

Theorem (Antonelli-B.-Fogagnolo-Pozzetta)

If (M^n, g) is non-collapsed and $Ric \geq 0$ then $I_M^{\frac{n}{n-1}}$ is concave. In particular I_M is strictly subadditive.

$$I_M(V) \leq I_M(|E^0|) + \ldots + I_M(|E^m|) \leq \mathrm{Per}(E^0) + \ldots + \mathrm{Per}(E^m) \leq I_M(V)$$

Corollary

There exists $0 \leq i \leq m$ such that $E^j = \emptyset$ for $j \neq i$.

[Bavard, Pansu '86], [Bayle '03], [Mondino, Nardulli '16], [Antonelli, B., Fogagnolo, Pozzetta '21].

- If $i = 0$, then we have done
- If not, we assume wlog $E_k \to E^1 \subset Y_1$. We have

$$|E^1| = V\,, \quad \mathrm{Per}(E^1) \le I_M(V)\,.$$

- Sharp isoperimetric inequality:

$$\mathrm{Per}(E^1) \ge n\omega_n^{\frac{1}{n}} \mathrm{AVR}(Y_1)^{\frac{1}{n}} |E_1|^{\frac{n-1}{n}} = n\omega_n^{\frac{1}{n}} \mathrm{AVR}(Y_1)^{\frac{1}{n}} V^{\frac{n-1}{n}}\,.$$

- Assumption II:

$$\mathrm{Per}(E^1) \ge n\omega_n^{\frac{1}{n}} \mathrm{AVR}(Y_1)^{\frac{1}{n}} V^{\frac{n-1}{n}} \ge n\omega_n^{\frac{1}{n}} (\mathrm{AVR}(g) + \varepsilon)^{\frac{1}{n}} V^{\frac{n-1}{n}}\,,$$

- Conclusion:

$$I_M(V) \ge n\omega_n^{\frac{1}{n}} (\mathrm{AVR}(g) + \varepsilon)^{\frac{1}{n}} V^{\frac{n-1}{n}}\,,$$

false for V big enough! The ball $B_R(p) \subset M$ is a better competitor when $R \to \infty$.

- If $i = 0$, then we have done
- If not, we assume wlog $E_k \to E^1 \subset Y_1$. We have

$$|E^1| = V\,, \quad \mathrm{Per}(E^1) \leq I_M(V)\,.$$

- Sharp isoperimetric inequality:

$$\mathrm{Per}(E^1) \geq n\,\omega_n^{\frac{1}{n}}\,\mathrm{AVR}(Y_1)^{\frac{1}{n}}|E_1|^{\frac{n-1}{n}} = n\,\omega_n^{\frac{1}{n}}\,\mathrm{AVR}(Y_1)^{\frac{1}{n}}\,V^{\frac{n-1}{n}}\,.$$

- Assumption II:

$$\mathrm{Per}(E^1) \geq n\,\omega_n^{\frac{1}{n}}\,\mathrm{AVR}(Y_1)^{\frac{1}{n}}\,V^{\frac{n-1}{n}} \geq n\,\omega_n^{\frac{1}{n}}(\mathrm{AVR}(g)+\varepsilon)^{\frac{1}{n}}\,V^{\frac{n-1}{n}}\,,$$

- Conclusion:

$$I_M(V) \geq n\,\omega_n^{\frac{1}{n}}(\mathrm{AVR}(g)+\varepsilon)^{\frac{1}{n}}\,V^{\frac{n-1}{n}}\,,$$

false for V big enough! The ball $B_R(p) \subset M$ is a better competitor when $R \to \infty$.

- If $i = 0$, then we have done
- If not, we assume wlog $E_k \to E^1 \subset Y_1$. We have

$$|E^1| = V\,, \quad \mathrm{Per}(E^1) \le I_M(V)\,.$$

- Sharp isoperimetric inequality:

$$\mathrm{Per}(E^1) \ge n\,\omega_n^{\frac{1}{n}}\mathrm{AVR}(Y_1)^{\frac{1}{n}}|E_1|^{\frac{n-1}{n}} = n\,\omega_n^{\frac{1}{n}}\mathrm{AVR}(Y_1)^{\frac{1}{n}}V^{\frac{n-1}{n}}\,.$$

- Assumption II:

$$\mathrm{Per}(E^1) \ge n\,\omega_n^{\frac{1}{n}}\mathrm{AVR}(Y_1)^{\frac{1}{n}}V^{\frac{n-1}{n}} \ge n\,\omega_n^{\frac{1}{n}}(\mathrm{AVR}(g)+\varepsilon)^{\frac{1}{n}}V^{\frac{n-1}{n}}\,,$$

- Conclusion:

$$I_M(V) \ge n\,\omega_n^{\frac{1}{n}}(\mathrm{AVR}(g)+\varepsilon)^{\frac{1}{n}}V^{\frac{n-1}{n}}\,,$$

false for V big enough! The ball $B_R(p) \subset M$ is a better competitor when $R \to \infty$.

- If $i = 0$, then we have done
- If not, we assume wlog $E_k \to E^1 \subset Y_1$. We have

$$|E^1| = V\,, \quad \mathrm{Per}(E^1) \le I_M(V)\,.$$

- Sharp isoperimetric inequality:

$$\mathrm{Per}(E^1) \ge n\omega_n^{\frac{1}{n}} \mathrm{AVR}(Y_1)^{\frac{1}{n}} |E_1|^{\frac{n-1}{n}} = n\omega_n^{\frac{1}{n}} \mathrm{AVR}(Y_1)^{\frac{1}{n}} V^{\frac{n-1}{n}}\,.$$

- Assumption II:

$$\mathrm{Per}(E^1) \ge n\omega_n^{\frac{1}{n}} \mathrm{AVR}(Y_1)^{\frac{1}{n}} V^{\frac{n-1}{n}} \ge n\omega_n^{\frac{1}{n}} (\mathrm{AVR}(g)+\varepsilon)^{\frac{1}{n}} V^{\frac{n-1}{n}}\,,$$

- Conclusion:

$$I_M(V) \ge n\omega_n^{\frac{1}{n}} (\mathrm{AVR}(g)+\varepsilon)^{\frac{1}{n}} V^{\frac{n-1}{n}}\,,$$

false for V big enough! The ball $B_R(p) \subset M$ is a better competitor when $R \to \infty$.

- If $i = 0$, then we have done
- If not, we assume wlog $E_k \to E^1 \subset Y_1$. We have

$$|E^1| = V\,, \quad \mathrm{Per}(E^1) \le I_M(V)\,.$$

- Sharp isoperimetric inequality:

$$\mathrm{Per}(E^1) \ge n\omega_n^{\frac{1}{n}} \mathrm{AVR}(Y_1)^{\frac{1}{n}} |E_1|^{\frac{n-1}{n}} = n\omega_n^{\frac{1}{n}} \mathrm{AVR}(Y_1)^{\frac{1}{n}} V^{\frac{n-1}{n}}\,.$$

- Assumption II:

$$\mathrm{Per}(E^1) \ge n\omega_n^{\frac{1}{n}} \mathrm{AVR}(Y_1)^{\frac{1}{n}} V^{\frac{n-1}{n}} \ge n\omega_n^{\frac{1}{n}} (\mathrm{AVR}(g) + \varepsilon)^{\frac{1}{n}} V^{\frac{n-1}{n}}\,,$$

- Conclusion:

$$I_M(V) \ge n\omega_n^{\frac{1}{n}} (\mathrm{AVR}(g) + \varepsilon)^{\frac{1}{n}} V^{\frac{n-1}{n}}\,,$$

false for V big enough! The ball $B_R(p) \subset M$ is a better competitor when $R \to \infty$.

- If $i = 0$, then we have done
- If not, we assume wlog $E_k \to E^1 \subset Y_1$. We have

$$|E^1| = V\,, \quad \mathrm{Per}(E^1) \le I_M(V)\,.$$

- Sharp isoperimetric inequality:

$$\mathrm{Per}(E^1) \ge n\omega_n^{\frac{1}{n}} \mathrm{AVR}(Y_1)^{\frac{1}{n}} |E_1|^{\frac{n-1}{n}} = n\omega_n^{\frac{1}{n}} \mathrm{AVR}(Y_1)^{\frac{1}{n}} V^{\frac{n-1}{n}}\,.$$

- Assumption II:

$$\mathrm{Per}(E^1) \ge n\omega_n^{\frac{1}{n}} \mathrm{AVR}(Y_1)^{\frac{1}{n}} V^{\frac{n-1}{n}} \ge n\omega_n^{\frac{1}{n}} (\mathrm{AVR}(g) + \varepsilon)^{\frac{1}{n}} V^{\frac{n-1}{n}}\,,$$

- Conclusion:

$$I_M(V) \ge n\omega_n^{\frac{1}{n}} (\mathrm{AVR}(g) + \varepsilon)^{\frac{1}{n}} V^{\frac{n-1}{n}}\,,$$

  false for V big enough! The ball $B_R(p) \subset M$ is a better competitor when $R \to \infty$.

Is our assumption sharp?

[Kasue and Washio, '90]: there exists a metric g on $\mathbb{R}^n$, $n \geq 4$ such that:

- $\mathrm{Ric}_g \geq 0$ and $(\mathbb{R}^n, g)$ does not split any line;
- $(\mathbb{R}^n, g)$ has Euclidean volume growth and the tangent cone at infinity is $\mathbb{R} \times C(\mathbb{S}_r^{n-2})$;
- $(\mathbb{R}^n, g)$ does not satisfy our assumption;
- $(\mathbb{R}^n, g)$ admits isoperimetric regions of any volume.

Question

Does any manifold with nonnegative Ricci curvature and Euclidean volume growth admit isoperimetric regions of large volume?

Is our assumption sharp?

[Kasue and Washio, '90]: there exists a metric g on $\mathbb{R}^n$, $n \geq 4$ such that:

- $\mathrm{Ric}_g \geq 0$ and $(\mathbb{R}^n, g)$ does not split any line;
- $(\mathbb{R}^n, g)$ has Euclidean volume growth and the tangent cone at infinity is $\mathbb{R} \times C(\mathbb{S}^{n-2}_r)$;
- $(\mathbb{R}^n, g)$ does not satisfy our assumption;
- $(\mathbb{R}^n, g)$ admits isoperimetric regions of any volume.

Question

Does any manifold with nonnegative Ricci curvature and Euclidean volume growth admit isoperimetric regions of large volume?

Is our assumption sharp?

[Kasue and Washio, '90]: there exists a metric g on $\mathbb{R}^n$, $n \geq 4$ such that:

- $\mathrm{Ric}_g \geq 0$ and $(\mathbb{R}^n, g)$ does not split any line;
- $(\mathbb{R}^n, g)$ has Euclidean volume growth and the tangent cone at infinity is $\mathbb{R} \times C(\mathbb{S}^{n-2}_r)$;
- $(\mathbb{R}^n, g)$ does not satisfy our assumption;
- $(\mathbb{R}^n, g)$ admits isoperimetric regions of any volume.

Question

Does any manifold with nonnegative Ricci curvature and Euclidean volume growth admit isoperimetric regions of large volume?

Is our assumption sharp?

[Kasue and Washio, '90]: there exists a metric g on $\mathbb{R}^n$, $n \geq 4$ such that:

- $\mathrm{Ric}_g \geq 0$ and $(\mathbb{R}^n, g)$ does not split any line;
- $(\mathbb{R}^n, g)$ has Euclidean volume growth and the tangent cone at infinity is $\mathbb{R} \times C(\mathbb{S}^{n-2}_r)$;
- $(\mathbb{R}^n, g)$ does not satisfy our assumption;
- $(\mathbb{R}^n, g)$ admits isoperimetric regions of any volume.

Question

Does any manifold with nonnegative Ricci curvature and Euclidean volume growth admit isoperimetric regions of large volume?

Is our assumption sharp?

[Kasue and Washio, '90]: there exists a metric g on $\mathbb{R}^n$, $n \geq 4$ such that:

- $\mathrm{Ric}_g \geq 0$ and $(\mathbb{R}^n, g)$ does not split any line;
- $(\mathbb{R}^n, g)$ has Euclidean volume growth and the tangent cone at infinity is $\mathbb{R} \times C(\mathbb{S}^{n-2}_r)$;
- $(\mathbb{R}^n, g)$ does not satisfy our assumption;
- $(\mathbb{R}^n, g)$ admits isoperimetric regions of any volume.

Question

Does any manifold with nonnegative Ricci curvature and Euclidean volume growth admit isoperimetric regions of large volume?

Is our assumption sharp?

[Kasue and Washio, '90]: there exists a metric g on $\mathbb{R}^n$, $n \geq 4$ such that:

- $\mathrm{Ric}_g \geq 0$ and $(\mathbb{R}^n, g)$ does not split any line;
- $(\mathbb{R}^n, g)$ has Euclidean volume growth and the tangent cone at infinity is $\mathbb{R} \times C(\mathbb{S}_r^{n-2})$;
- $(\mathbb{R}^n, g)$ does not satisfy our assumption;
- $(\mathbb{R}^n, g)$ admits isoperimetric regions of any volume.

Question

Does any manifold with nonnegative Ricci curvature and Euclidean volume growth admit isoperimetric regions of large volume?

Open questions

- Can we describe Isoperimetric sets of big volume?
- How do they relate to cross-sections of cones at infinity?
- Under our assumption, are isoperimetric set of big volume unique? Stable? Do they foliate the space?

Thank you for your attention!

Open questions

- Can we describe Isoperimetric sets of big volume?
- How do they relate to cross-sections of cones at infinity?
- Under our assumption, are isoperimetric set of big volume unique? Stable? Do they foliate the space?

Thank you for your attention!

Open questions

- Can we describe Isoperimetric sets of big volume?
- How do they relate to cross-sections of cones at infinity?
- Under our assumption, are isoperimetric set of big volume unique? Stable? Do they foliate the space?

Thank you for your attention!