Lecture 5.
Sharp differential inequality for the isoperimetric profile on noncollapsed spaces with lower Ricci bounds, and sharp and rigid isoperimetric inequality for nonnegatively curved spaces

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Sharp differential inequality for the isoperimetric profile

The statement

Theorem (A.–Pasqualetto–Pozzetta–Semola)

Let $(X, d, \mathcal{H}^N)$ *be an* $\mathrm{RCD}(K, N)$ *space, with* $K \in \mathbb{R}$ *and* $2 \leq N < +\infty$. *Assume that there exists* $v_0 > 0$ *such that* $\mathcal{H}^N(B_1(x)) \geq v_0$ *for every* $x \in X$.
Let $I : (0, \mathcal{H}^N(X)) \to (0, \infty)$ *be the isoperimetric profile of* X. *Then the inequality*

$$-I''I \geq K + \frac{(I')^2}{N-1} \quad \text{holds in the viscosity sense on } (0, \mathcal{H}^N(X)),$$

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⋆ Viscosity: for all C^2 φ's touching f at x_0, with $\varphi \leq f$ around x_0, inequality holds with φ.

⋆ Sharp. Equality in the models.

⋆ In the compact case in [Bavard–Pansu, Bayle]. In the noncompact case challenging, see, e.g., Euclidean convex bodies with bounded geometry in [Leonardi–Ritoré–Vernadakis].

The proof in the smooth compact case (I)

⋆ Assume X is an N-dimensional compact manifold with $\mathrm{Ric} \geq K$.
Main ingredients: existence of isoperimetric sets, and first and second variation of the area (with regularity theory for isoperimetric boundaries).

The proof in the smooth compact case (I)

$\star$ Assume X is an N-dimensional compact manifold with $\mathrm{Ric} \geq K$.
Main ingredients: existence of isoperimetric sets, and first and second variation of the area (with regularity theory for isoperimetric boundaries).

Step 1. Fix an isoperimetric region E of volume v. For simplicity ∂E smooth, and $I'(v), I''(v)$ exist. Denote ν the unit normal to ∂E, II the second fundamental form, and H the mean curvature.

Step 2. If E_t is the t-enlargment, with $t \in (-\varepsilon, \varepsilon)$, then

$$\frac{\mathrm{d}}{\mathrm{d}t}|_{t=0}\mathrm{Per}(E_t) = \int_{\partial E} H = H\mathrm{Per}(E)$$

$$\frac{\mathrm{d}^2}{\mathrm{d}t^2}|_{t=0}\mathrm{Per}(E_t) = \int_{\partial E} \left(H^2 - |II|^2 - \mathrm{Ric}(\nu,\nu)\right) \leq \left(\frac{N-2}{N-1}H^2 - K\right)\mathrm{Per}(E)$$

The proof in the smooth compact case (II)

Step 3. From the definition $I(\mathrm{vol}(E_t)) \leq \mathrm{Per}(E_t)$ for $t \in (-\varepsilon, \varepsilon)$. Taking the first derivative at $t = 0$

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and taking the second derivative at $t = 0$,

$$I''(v)\mathrm{Per}(E)^2 + I'(v)H\mathrm{Per}(E) \leq \frac{\mathrm{d}^2}{\mathrm{d}t^2}|_{t=0}\mathrm{Per}(E_t) \leq \left(\frac{N-2}{N-1}H^2 - K\right)\mathrm{Per}(E),$$

from which, using $I(v) = \mathrm{Per}(E)$, we have

$$I''(v)I(v) + \frac{1}{N-1}I'(v)^2 + K \leq 0.$$

Noncompact case (I): dealing with possible non-existence

Theorem (A.–Nardulli–Pozzetta after A.–Fogagnolo–Pozzetta)

Let $(X, d, \mathcal{H}^N)$ *be a noncompact* $\mathrm{RCD}(K, N)$ *space, with* $K \in \mathbb{R}$ *and* $N \geq 2$*.*
Assume $\mathcal{H}^N(B_1(x)) \geq v_0 > 0$ *for every* $x \in X$ *and some* $v_0 > 0$*.*
Let $\{\Omega_i\}_{i \in \mathbb{N}}$ *be a minimizing sequence for the isoperimetric problem at volume* $V > 0$*.*

Noncompact case (I): dealing with possible non-existence

Theorem (A.–Nardulli–Pozzetta after A.–Fogagnolo–Pozzetta)

Let $(X, d, \mathcal{H}^N)$ be a noncompact $\mathrm{RCD}(K, N)$ space, with $K \in \mathbb{R}$ and $N \geq 2$. Assume $\mathcal{H}^N(B_1(x)) \geq v_0 > 0$ for every $x \in X$ and some $v_0 > 0$. Let $\{\Omega_i\}_{i \in \mathbb{N}}$ be a minimizing sequence for the isoperimetric problem at volume $V > 0$. Then, up to subsequences in i, there is $\ell \in \mathbb{N}$ such that the following hold:

- $\Omega_i = \Omega_i^c \sqcup \Omega_i^d$, *with Ω_i^c converging in L^1_{loc}, volume, and perimeter to an isoperimetric set Ω, and Ω_i^d diverging at infinity;*
- *For every $i \in \mathbb{N}$ and $1 \leq j \leq \ell$ there exist points $p_{i,j} \in X$ and disjoint $p_{i,j} \in \Omega_{i,j}^d \subset \Omega_i^d$ such that, for every $1 \leq j \leq \ell$, the sequences of points $\{p_{i,j}\}_{i \in \mathbb{N}}$ are mutually (wrt j) diverging;*
- *for every $1 \leq j \leq \ell$, $(X, d, \mathcal{H}^N, p_{i,j})$ pmGH conv. to $(X_j, \mathrm{d}_j, \mathcal{H}^N, p_j)$ and $\Omega_{i,j}^d$ converge to $Z_j \subset X_j$ in L^1-strong (in volume + $\chi_{\Omega_{i,j}^d}\mathcal{H}^N \rightharpoonup \chi_{Z_j}\mathcal{H}^N$) and perimeter, and Z_j is isoperimetric for its volume in X_j;*
- $V = \mathcal{H}^N(\Omega) + \sum_{j=1}^{\ell} \mathcal{H}^N(Z_j)$, *and* $I(V) = \mathrm{Per}(\Omega) + \sum_{j=1}^{\ell} \mathrm{Per}_{X_j}(Z_j)$.

Noncompact case (II): adapting variations of the area

Theorem (A.–Pasqualetto–Pozzetta–Semola)

Let $(X, d, \mathcal{H}^N)$ *be* $\mathrm{RCD}(0, N)$, $E \subset X$ *be an isop.region. Then, for some* $c \in \mathbb{R}$,

$$\Delta d_{\overline{E}} \geq \frac{c}{1 + \frac{c}{N-1} d_{\overline{E}}}, \quad \text{on } E$$

$$\Delta d_{\overline{E}} \leq \frac{c}{1 + \frac{c}{N-1} d_{\overline{E}}} \quad \text{on } X \setminus \overline{E}.$$

where $d_{\overline{E}}$ *is the signed distance function from* E.

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where $d_{\overline{E}}$ *is the signed distance function from* E. *In addition, being* E_t *the* t*-enlargement for* $t \in (-\varepsilon, \varepsilon)$, *we have*

$$\mathrm{Per}(E_t) \leq \left(1 + \frac{c}{N-1} t\right)^{N-1} \mathrm{Per}(E)$$

If $K \in \mathbb{R}$, the analogous inequality yields that if $t \mapsto \mathrm{Per}(E_t)$ is twice differentiable at $t = 0$, then $\frac{\mathrm{d}}{\mathrm{d}t}|_{t=0}\mathrm{Per}(E_t) = c\mathrm{Per}(E)$ and $\frac{\mathrm{d}^2}{\mathrm{d}t^2}|_{t=0}\mathrm{Per}(E_t) \leq \left(\frac{N-2}{N-1}c^2 - K\right)\mathrm{Per}(E)$.

Noncompact case (III): proof

Step 1. Fix $v > 0$ and take $\Omega, Z_1, \ldots, Z_\ell$ as in the asymptotic mass decomposition when minimizing at volume v. Take $\varphi \in C^2(v-\varepsilon, v+\varepsilon)$ with $\varphi \leq I$ and $\varphi(v) = I(v)$.

From $I(v) = \mathrm{Per}(\Omega) + \sum_{i=1}^{\ell} \mathrm{Per}(Z_i)$, the regularity of φ, and the previous inequality on $\mathrm{Per}(\Omega_t), \mathrm{Per}((Z_i)_t)$ we first get that $t \mapsto \mathrm{Per}(\Omega_t)$, and $t \mapsto \mathrm{Per}((Z_i)_t)$ are differentiable at $t = 0$, and we may choose the same c for all the sets $\Omega, Z_1, \ldots, Z_\ell$;

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Step 2. Calling $\beta(t) := \mathcal{H}^N(\Omega_t) + \sum_{i=1}^{\ell} \mathcal{H}^N((Z_i)_t)$ we have for $t \in (-\varepsilon, \varepsilon)$

$$\varphi(\beta(t)) \leq I(\beta(t)) \leq \mathrm{Per}(\Omega_t) + \sum_{i=1}^{\ell} \mathrm{Per}((Z_i)_t) \leq \left(1 + \frac{c}{N-1} t\right)^{N-1} I(v),$$

and taking the second derivative in $t = 0$ gives the sought conclusion.

(Some) consequences of the sharp differential inequality

Differentiability properties of the isoperimetric profile

Corollary (A.–Pasqualetto–Pozzetta–Semola)

Let $(X, d, \mathcal{H}^N)$ be an $\mathrm{RCD}(K, N)$ space, with $K \in \mathbb{R}$ and $2 \leq N < +\infty$. Assume that there exists $v_0 > 0$ such that $\mathcal{H}^N(B_1(x)) \geq v_0$ for every $x \in X$.

Let $I : (0, \mathcal{H}^N(X)) \to (0, \infty)$ be the isoperimetric profile of X. The following hold:

- *I is locally C-concave. Namely, for every $v \in (0, \mathcal{H}^N(X))$ and for every $\delta \in (0, \mathcal{H}^N(X) - v)$ there exists $C > 0$ such that the function $I(x) - Cx^2$ is concave on $(v - \delta, v + \delta)$;*
- *I has right and left derivatives defined for every $v \in (0, \mathcal{H}^N(X))$;*
- *I is differentiable in a co-countable subset of $(0, \mathcal{H}^N(X))$, it is locally Lipschitz, it is twice differentiable almost-everywhere, and the Sharp Differential Inequality for I holds pointwise almost everywhere.*

(Some) concavity properties of the isoperimetric profile

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Let $I : (0, \mathcal{H}^N(X)) \to (0, \infty)$ *be the isoperimetric profile of* X*, and let* $\psi := I^{\frac{N}{N-1}}$. *The the following hold:*

- *The inequality*

$$-\psi'' \geq \frac{KN}{N-1}\psi^{\frac{2-N}{N}} \text{ holds in the visc. sense and a.e. on } (0, \mathcal{H}^N(X));$$

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- *The quantity* $v \mapsto I(v)/v^{\frac{N-1}{N}}$ *has limit as* $v \to 0$.

Isoperimetric sets and connectedness

Corollary (A.–Pasqualetto–Pozzetta–Semola)

Let $(X, d, \mathcal{H}^N)$ be an $\mathrm{RCD}(0, N)$ space, with $2 \leq N < +\infty$. Assume that there exists $v_0 > 0$ such that $\mathcal{H}^N(B_1(x)) \geq v_0$ for every $x \in X$. Let $E \subset X$ be an isoperimetric set. Then E is connected.

Proof. We have

$$(I^{\frac{N}{N-1}})'' \leq 0,$$

hence I is strictly subadditive.

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$$I(\mathcal{H}^N(E)) < I(\mathcal{H}^N(E_1)) + I(\mathcal{H}^N(E_2)) \leq \mathrm{Per}(E).$$

From indecomposability to conn. use [Bonicatto–Pasqualetto–Rajala].

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⋆ For arbitrary lower bounds K the same holds for small volumes.

Lipschitz properties of the isoperimetric profile

Corollary (A.–Pasqualetto–Pozzetta–Semola)

Let $0 < V_1 < V_2 < V_3$, *and let* $K \in \mathbb{R}, N \geq 2, v_0 > 0$. *Then there exists* $L, \Lambda, R > 0$ *depending on* K, N, v_0, V_1, V_2, V_3 *such that:*

- *If* $(X, d, \mathcal{H}^N)$ *is* $\mathrm{RCD}(K, N)$ *with* $\inf_{x \in X} \mathcal{H}^N(B_1(x)) \geq v_0 > 0$ *and* $\mathcal{H}^N(X) \geq V_3$, *then*

$$v \mapsto I(v) \text{ is } L\text{-Lipschitz on } [V_1, V_2] .$$

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$$v \mapsto I(v) \text{ is } L\text{-Lipschitz on } [V_1, V_2] \,.$$

- *If* $E \subset X$ *is an isoperimetric region with* $\mathcal{H}^N(E) \in [V_1, V_2]$, *then* E *is a* (Λ, R)*-minimizer, i.e., for any* $F \subset X$ *such that* $F \Delta E \subset B_R(x)$ *for some* $x \in X$, *then*

$$\mathrm{Per}(E) \leq \mathrm{Per}(F) + \Lambda \mathcal{H}^N(F \Delta E) \,.$$

⋆ Uniform density estimates on isoperimetric boundaries.

Sharp and rigid isoperimetric inequality for nonnegatively curved spaces

Sharp isoperimetric inequality in nonnegative curvature

Theorem (Balogh–Kristaly)

Let $(X, \mathrm{d}, \mathfrak{m})$ *be a* $\mathrm{CD}(0, N)$ *space. Let us assume*

$$\mathrm{AVR}(X, \mathsf{d}, \mathfrak{m}) := \lim_{r \to +\infty} \frac{\mathfrak{m}(B_r(x))}{\omega_N r^N} > 0,$$

for $x \in X$, ω_N *being the volume of the unit ball in* $\mathbb{R}^N$. *Hence for every set* E *of finite perimeter in* X, *it holds*

$$\mathrm{Per}(E) \geq N(\omega_N \mathrm{AVR})^{\frac{1}{N}} \mathfrak{m}(E)^{\frac{N-1}{N}}.$$

Sharp isoperimetric inequality in nonnegative curvature

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for $x \in X$, ω_N being the volume of the unit ball in $\mathbb{R}^N$. Hence for every set E of finite perimeter in X, it holds

$$\mathrm{Per}(E) \geq N(\omega_N \mathrm{AVR})^{\frac{1}{N}} \mathfrak{m}(E)^{\frac{N-1}{N}}.$$

Moreover if X is a smooth Riemannian manifold and E has a C^1 boundary (always if $N \leq 7$), equality above holds if and only if X is isometric to $\mathbb{R}^N$ and E is isometric to a ball.

$\star$ Sharp because reached on balls with radius $R \to +\infty$.

Brief overview of the literature

(i) [Agostiniani–Fogagnolo–Mazzieri] $\to$ Proof of the sharp and rigid inequality for 3-dimensional Riemannian manifolds;

(ii) [Brendle] $\to$ Proof of the sharp inequality for Riemannian manifolds, and rigidity assuming regularity of the isoperimetric boundary.
[Fogagnolo–Mazzieri] $\to$ Proof of sharp and rigid inequality for n-dimensional Riemannian manifolds with $n \leq 7$.
[Johne] Proof of the sharp inequality for manifolds with density and non-negative Bakry-Émery Ricci curvature;

(iii) [Balogh–Kristály] Proof of the sharp inequality for $\mathrm{CD}(0, N)$ spaces and rigidity for Riemannian manifolds assuming regularity of the isoperimetric boundary;

(iv) [Cavalletti–Manini] Extension of the sharp inequality to $\mathrm{MCP}(0, N)$ spaces.

Metric cones

Definition (Metric cone)

Let (X, d) be a metric space.
Let $C(X) := [0, +\infty) \times X / \{0\} \times X$, and for $(t, p), (s, q) \in C(X)$ consider

$$d_c((t, p), (s, q)) := \sqrt{t^2 + s^2 - 2ts \cos(\min\{d(p, q), \pi\})}.$$

We say that $(C(X), d_c)$ is the *metric cone* over (X, d). The collapsed point relative to $\{0\} \times X$ is called a *tip* of the cone.

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Remark. From [Ketterer] we have that if $N \geq 2$, $(C(Z), d_c, \mathcal{H}^N)$ is an $\mathrm{RCD}(0, N)$ space if and only if $(Z, d, \mathcal{H}^{N-1})$ is an $\mathrm{RCD}(N-2, N-1)$ space with $\mathrm{diam}(Z) \leq \pi$.
An arbitrary $\mathrm{RCD}(0, N)$ space $(C(Z), d_c, \mathcal{H}^N)$ will be called *Euclidean metric measure cone of dimension* N.

Rigidity for N-dimensional $\mathrm{RCD}(0, N)$ spaces

Theorem (A.–Pasqualetto–Pozzetta–Semola)

Let $(X, d, \mathcal{H}^N)$ be an $\mathrm{RCD}(0, N)$ metric measure space, for some $N \geq 2$, with $\mathrm{AVR}(X, d, \mathcal{H}^N) > 0$.
Then a set E with $\mathcal{H}^N(E) \in (0, \infty)$ reaches equality in the sharp isoperimetric inequality if and only if

- *X is isometric to a Euclidean metric measure cone of dimension N,*
- *and E is isometric to a ball centered at one of the tips of X.*

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- *X is isometric to a Euclidean metric measure cone of dimension N,*
- *and E is isometric to a ball centered at one of the tips of X.*

⋆ No assumptions on the regularity of ∂E,
⋆ If X is a Riemannian manifold, it gives the rigidity with the model $\mathbb{R}^n$ without assuming the regularity of ∂E.

Proof (I): Rigid upper bound on the mean curvature barrier

Recall. In the hypotheses, $E \subset X$ is an isoperimetric region. Then, for some $c \in \mathbb{R}$,

$$\Delta d_{\overline{E}} \geq \frac{c}{1 + \frac{c}{N-1} d_{\overline{E}}} \quad \text{on } E$$

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Proof (I): Rigid upper bound on the mean curvature barrier

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From the second $c \geq 0$, and by the first $r_E := \sup_{x \in E} d(x, X \setminus E) \leq \frac{N-1}{c}$.

Proposition (A.–Pasqualetto–Pozzetta–Semola)

If $r_E = \frac{N-1}{c}$, then E is isometric to a ball of radius r_E in a tip of a Euclidean metric measure cone of dimension N.

Idea of proof. Take x_0 reaching maximum, consider $d_{\overline{E}} + d_{x_o}$.

Proof (II): Rigid upper bound on the mean curv. barrier

Proposition (A.–Pasqualetto–Pozzetta–Semola)

For the isoperimetric region $E \subset X$ in the hypotheses, and for (one of) its mean curvature barrier(s) c, we have

$$c \le \frac{N-1}{N}\frac{\mathrm{Per}(E)}{\mathcal{H}^N(E)},$$

and the equality holds if and only if E is isometric to a ball of radius $\frac{N-1}{c}$ in a tip of a Euclidean metric measure cone of dimension N.

Proof (II): Rigid upper bound on the mean curv. barrier

Proposition (A.–Pasqualetto–Pozzetta–Semola)

For the isoperimetric region $E \subset X$ in the hypotheses, and for (one of) its mean curvature barrier(s) c, we have

$$c \le \frac{N-1}{N} \frac{\mathrm{Per}(E)}{\mathcal{H}^N(E)},$$

and the equality holds if and only if E is isometric to a ball of radius $\frac{N-1}{c}$ in a tip of a Euclidean metric measure cone of dimension N.

Proof. From co-area and the perimeter estimate of the enlargements

$$\mathcal{H}^N(\{x \in E : d(x, X \setminus E) \le r\}) \le \mathrm{Per}(E) \int_0^r \left(1 - \frac{c}{N-1} s\right)^{N-1} ds$$

for every $r \le \frac{N-1}{c}$. Putting $r = \frac{N-1}{c}$ and using $r_E \le \frac{N-1}{c}$ we have

$$\mathcal{H}^N(E) \le \mathrm{Per}(E) \frac{N-1}{cN}, \quad \text{and the rigidity follows by the rigidity of } r_E.$$

Proof (III): Lower bound on the mean curvature barrier

Proposition (A.–Pasqualetto–Pozzetta–Semola)

For the isoperimetric region $E \subset X$ in the hypotheses, and for (one of) its mean curvature barrier(s) c, we have

$$c \geq (N-1)\left(\frac{N\omega_N \mathrm{AVR}(X)}{\mathrm{Per}(E)}\right)^{\frac{1}{N-1}}.$$

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Proposition (A.–Pasqualetto–Pozzetta–Semola)

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Proof. By co-area and the estimates of the perimeter of the enlargements

$$\mathcal{H}^N(\{x \in X \setminus E \,:\, d(x,E) \leq r\}) \leq \mathrm{Per}(E) \int_0^r \left(1 + \frac{c}{N-1} s\right)^{N-1} ds$$

$$= \mathrm{Per}(E) \frac{N-1}{Nc} \left[\left(1 + \frac{cr}{N-1}\right)^N - 1\right] \sim_{r \to +\infty} \mathrm{Per}(E) \frac{c^{N-1}}{N(N-1)^{N-1}} r^N .$$

This, together with $\mathcal{H}^N(\{x \in X \setminus E \,:\, d(x,E) \leq r\}) \sim_{r \to +\infty} \mathrm{AVR}(X)\omega_N r^N$, concludes the proof.

Proof (IV): End of the proof

Proof. We have

$$(\star)\quad c \geq (N-1)\left(\frac{N\omega_N \mathrm{AVR}(X)}{\mathrm{Per}(E)}\right)^{\frac{1}{N-1}}, \qquad (\star\star)\quad c \leq \frac{N-1}{N}\frac{\mathrm{Per}(E)}{\mathcal{H}^N(E)}.$$

Thus,

$$\mathrm{Per}(E) \geq \max\left\{\frac{N\mathcal{H}^N(E)c}{(N-1)}, \mathrm{AVR}(X)\omega_N \cdot \frac{N(N-1)^{N-1}}{c^{N-1}}\right\},$$

and then

$$\mathrm{Per}(E) \geq \left(\frac{N\mathcal{H}^N(E)c}{(N-1)}\right)^{\frac{N-1}{N}}\left(\mathrm{AVR}(X)\omega_N \cdot \frac{N(N-1)^{N-1}}{c^{N-1}}\right)^{\frac{1}{N}}$$
$$= N(\omega_N \mathrm{AVR})^{\frac{1}{N}}\mathcal{H}^N(E)^{\frac{N-1}{N}}.$$

Proof (IV): End of the proof

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$$= N(\omega_N \mathrm{AVR})^{\frac{1}{N}}\mathcal{H}^N(E)^{\frac{N-1}{N}}.$$

If $=$ holds, by ineq. on r_E we have $E \supset B_{\frac{N-1}{c}}(x)$, and since the LHS of $(\star), (\star\star)$ are equal, we have by Bishop–Gromov that $E = B$, and B saturates the AVR ratio. Hence by volume cone implies metric cone [De Philippis–Gigli] we get the conclusion.

An alternative proof of the sharp isoperimetric inequality

Remark. We proved the isoperimetric inequality on isoperimetric sets of $\mathrm{RCD}(0,N)$ spaces $(X,d,\mathcal{H}^N)$ with $\mathrm{AVR}>0$. By means of the asymptotic mass decomposition theorem (AMDT) we can thus recover the sharp isoperimetric inequality on arbitrary sets.

Take $E\subset X$ of finite perimeter, with $V:=\mathcal{H}^N(E)$. Take Ω_i a minim. sequence for volume V. In the setting of AMDT

$$
\begin{aligned}
\mathrm{Per}(E) \geq I(V) &= \mathrm{Per}(\Omega)+\sum_{j=1}^{\ell}\mathrm{Per}_{X_j}(Z_j) \\
&\geq N\omega_N^{\frac{1}{N}}\left(\mathrm{AVR}(X)^{\frac{1}{N}}\mathcal{H}^N(\Omega)^{\frac{N-1}{N}}+\sum_{j=1}^{\ell}\mathrm{AVR}(X_j)^{\frac{1}{N}}\mathcal{H}^N(Z_j)^{\frac{N-1}{N}}\right) \\
&\geq N(\omega_N\mathrm{AVR}(X))^{\frac{1}{N}}\left(\mathcal{H}^N(\Omega)+\sum_{j=1}^{\ell}\mathcal{H}^N(Z_j)\right)^{\frac{N-1}{N}} = N(\omega_N\mathrm{AVR}(X))^{\frac{1}{N}}V^{\frac{N-1}{N}}.
\end{aligned}
$$

Thank you for the attention