On Clusters and the Multi-Isoperimetric Profile in Riemannian Manifolds with Bounded Geometry

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Isoperimetric Problems (University of Pisa)

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Conteúdo

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Clusters

An **N-cluster** $\mathcal{E}$ of (M^n, g) is a finite family of sets of finite perimeter $\mathcal{E} := \{\mathcal{E}(h)\}_{h=1}^{N}$, $N \in \mathbb{N}$, $N \geq 1$, with

$$0 < \mathbf{Vol}_g\left(\mathcal{E}(h)\right) < +\infty, \qquad 1 \leq h \leq N,$$

$$\mathbf{Vol}_g\left(\mathcal{E}(h) \cap \mathcal{E}(k)\right) = 0, \qquad 1 \leq h < k \leq N.$$

The sets $\mathcal{E}(h)$ are called the **chambers** of $\mathcal{E}$. The **exterior chamber** of $\mathcal{E}$ is defined as

$$\mathcal{E}(0) = M^n \setminus \bigcup_{h=1}^{N} \mathcal{E}(h).$$

Clusters

In particular, $\{\mathcal{E}(h)\}_{h=0}^{N}$ is a partition of M^n (up to a set of null volume). The **volume vector** $\mathbf{v}_g(\mathcal{E})$ is defined as

$$\mathbf{v}_g(\mathcal{E}) = (\mathbf{Vol}_g(\mathcal{E}(1)), \ldots, \mathbf{Vol}_g(\mathcal{E}(N))) \in \mathbb{R}^N.$$

We let $\mathbb{R}_+^N$ be the set of those $\mathbf{v} \in \mathbb{R}^N$ such that $\mathbf{v}(h) > 0$ (the h-th component of a vector $\mathbf{v}$) for every $h = 1, \ldots, N$. Notice that if $\mathcal{E}$ is an N-cluster, then $\mathbf{v}_g(\mathcal{E}) \in (0, \mathbf{Vol}_g(M))^N \subset \mathbb{R}_+^N$ as $\mathbf{v}_g(\mathcal{E})(h) = \mathbf{Vol}_g(\mathcal{E}(h)) > 0$ for every $h = 1, \ldots, N$.

Clusters

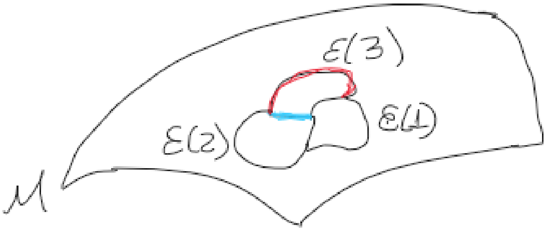
The **interfaces** of the N-cluster $\mathcal{E}$ in (M^n, g) are the $\mathcal{H}_g^{n-1}$-rectifiable sets

$$\mathcal{E}(h,k) = \partial^*\mathcal{E}(h) \cap \partial^*\mathcal{E}(k), \quad 0 \leq h,k \leq N, h \neq k.$$

We define the **relative perimeter of $\mathcal{E}$ in** $F \subset M^n$ as

$$\mathcal{P}_g\left(\mathcal{E}, F\right) = \sum_{1 \leq h < k \leq N} \mathcal{H}_g^{n-1}\left(F \cap \mathcal{E}(h,k)\right), \tag{1}$$

where F is any Borelian set in (M^n, g). The **perimeter of $\mathcal{E}$** is denoted $\mathcal{P}_g(\mathcal{E}) \doteq \mathcal{P}_g\left(\mathcal{E}, M\right)$.



$$\mathcal{E}(2,3) = \partial^* \mathcal{E}(2) \cap \partial^* \mathcal{E}(3)$$

$$\mathcal{E}(3,0) = \partial^* \mathcal{E}(3) \cap \partial^* \mathcal{E}(0)$$

The **flat distance in $F \subset M^n$ of two N-clusters $\mathcal{E}$ and $\mathcal{E}'$ of** (M^n, g) is defined as

$$d^F_{\mathcal{F},g}(\mathcal{E},\mathcal{E}') := \sum_{h=1}^{N} \mathbf{Vol}_g \left(F \cap (\mathcal{E}(h) \Delta \mathcal{E}'(h))\right).$$

We say that a sequence of N-clusters $\{\mathcal{E}_k\}_{k\in\mathbb{N}}$ in (M^n, g) **locally converges to** $\mathcal{E}$, and write $\mathcal{E}_k \overset{\text{loc}}{\to} \mathcal{E}$, if for every compact set $K \subset M^n$ we have $d^K_{\mathcal{F},g}(\mathcal{E},\mathcal{E}_k) \to 0$ as $k \to +\infty$. If $d_{\mathcal{F},g}(\mathcal{E},\mathcal{E}_k) \to 0$ as $k \to +\infty$, we say that $\mathcal{E}_k$ **converges to** $\mathcal{E}$ and we denote $\mathcal{E}_k \to \mathcal{E}$.

A really useful formula for the perimeter

Proposition - *Maggi, 2012*

If $\mathcal{E}$ is an N-cluster in (M^n, g), then for every $F \subset M^n$ we have

$$\mathcal{P}_g(\mathcal{E}; F) = \frac{1}{2} \sum_{h=0}^{N} \mathcal{P}_g(\mathcal{E}(h); F).$$

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In particular, if A is open in M^n and $\mathcal{E}_k \overset{loc}{\to} \mathcal{E}$, then

$$\mathcal{P}_g(\mathcal{E}; A) \leq \liminf_{k \to +\infty} \mathcal{P}_g(\mathcal{E}_k; A).$$

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The proof of this result is a straightforward adaptation of the Euclidean case.

Weighted perimeter

What happens if we define the perimeter as

$$\mathcal{P}_g^w(\mathcal{E}, F) = \frac{1}{2} \sum_{h,k=0}^{N} \alpha_{hk} \, \mathcal{H}_g^{n-1} \left(F \cap \mathcal{E}(h,k) \right),$$

where $\alpha_{hk} = \alpha_{kh} > 0$, and $\alpha_{hh} = 0$, for any $h, k \in \{0, \cdots, N\}$?

$\mathcal{E}(2)$  $\mathcal{E}(1)$

$\alpha_{hk} = 1 \quad \forall h \neq k$

arbitrary weights

$$\frac{1}{2}\sum_{h=0}^{2} P(\mathcal{E}(h)) = \frac{1}{2}\Big(P(\mathcal{E}(0)) + P(\mathcal{E}(1)) + P(\mathcal{E}(2)) \Big) \neq P(\mathcal{E}) = \frac{1}{2}\sum_{h,k=0}^{2} \alpha_{hk} H^{m-1}(\mathcal{E}(h,k))$$

$$= \frac{1}{2}\Big(2H^{m-1}(\mathcal{E}(2,0)) + 2H^{m-1}(\mathcal{E}(1,0)) + 2H^{m-1}(\mathcal{E}(1,2)) \Big)$$

$$= \frac{1}{2}\Big(2\alpha_{02} H^{m-1}(\mathcal{E}(0,2)) + 2\alpha_{01} H^{m-1}(\mathcal{E}(0,1)) + 2\alpha_{12} H^{m-1}(\mathcal{E}(1,2)) \Big)$$

An abstract condition

F. Almgren Jr in 1975 introduced the following condition.

Partitioning regular coefficients matrices

1. $\alpha_{ij} = \alpha_{ji} > 0$ and $\alpha_{ii} = 0$ for any $i, j \in \{1, \cdots, N\}$;
2. for each $i \in \{1, \cdots, N\}$ and each vector $a = (a_1, \cdots, a_N) \in \mathbb{R}_+^N$ such that $a_k > 0$ for some $k \neq i$, there exists $j \in \{1, \cdots, N\} \setminus \{i\}$ such that

$$a_j \alpha_{ij} > \sum_{k=1, k \neq i,j}^{N} a_k \alpha_{jk}.$$

The strict triangle inequality

B. White has introduced the following condition on the coefficients

$$\alpha_{hk} \leq \alpha_{h\ell} + \alpha_{\ell k},$$

for any $h, k, \ell \in \{0, \cdots, N\}$ with $\ell \notin \{h, k\}$.

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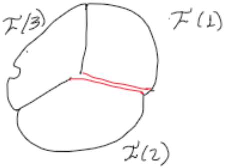

$\mathcal{V}(\mathcal{E}) = \mathcal{V}(\mathcal{F})$

$\mathcal{H}^{n-1}(\mathcal{E}(1,2)) \approx \mathcal{H}^{n-1}(\mathcal{F}(1,3) \cap A)$
$\approx \mathcal{H}^{n-1}(\mathcal{F}(2,3) \cap A)$

$\Longrightarrow P(\mathcal{F}) < P(\mathcal{E})$

if $\alpha_{12} > \alpha_{13} + \alpha_{23}$

- B. White in 1996,
- Ambrosio and Braides in 1990.

Theorem

It holds that $\alpha_{hk} \leq \alpha_{h\ell} + \alpha_{\ell k}$ for any $\ell \in \{0, \cdots, N\} \setminus \{h, k\}$ if, and only if, the perimeter functional is lower semicontinuous.

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- B. White proved the existence of minimizers in 1996.

Theorem - *White, 1996*

If the coefficients α_{hk} satisfy $\alpha_{hk} = \alpha_{kh} > 0, \alpha_{hh} = 0$, for any $h, k \in \{0, \cdots, N\}$, and the strict triangle inequality, then there exists a weighted isoperimetric N-cluster in $\mathbb{R}^n$.

Consequences of the strict triangle inequality

- G. P. Leonardi prove the regularity of minimizers in 2001.

Theorem - *Leonardi, 2001*

If $\mathcal{E}$ is a weighted isoperimetric N-cluster in $\mathbb{R}^n$ and the coefficients satisfy the strict triangle inequality, then we have that the interfaces, up to a $\mathcal{H}^{n-1}$-null set, are made of smooth hypersurfaces with constant mean curvature.

Contents

The isoperimetric profile

Let (M^n, g) be a Riemannian manifold of dimension n. We define the **isoperimetric profile** as the function $I_{(M,g)} : (0, \mathbf{Vol}_g(M)) \to (0, +\infty)$ defined by

$$I_{(M,g)}(v) := \inf \{\mathcal{P}_g(E) : \quad E \subset M, \quad \mathbf{Vol}_g(E) = v\},$$

where $\mathcal{P}_g$ denotes the classical De Giorgi's perimeter.

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where $\mathcal{P}_g$ denotes the classical De Giorgi's perimeter.

- If $M = \mathbb{R}^n$ and $g = g_{euc}$, we know that $I_{(\mathbb{R}^n, g_{eucl})}(v) = c_n v^{\frac{n-1}{n}}$.

The isoperimetric profile

Theorem - *G. Antonelli, E. Pasqualetto, M. Pozzetta, and D. Semola, 2022*

Let $(X, \mathrm{d}, \mathcal{H}^n)$ be an RCD(κ, n) space with isoperimetric profile function I. Let us assume $\inf_{x \in X} \mathcal{H}^n(\mathbf{B}_g(x,1)) \geq v_0 > 0$. Then we have the following asymptotic for small volumes:

$$\lim_{v \to 0} \frac{I_X(v)}{v^{\frac{n-1}{n}}} = n\left(\omega_n \vartheta_{\infty,\min}\right)^{\frac{1}{n}}$$

where, being $v(n, \kappa/(n-1), r)$ the volume of the ball of mdius r in the simply connected model space with constant sectional curvature $\kappa/(n-1)$ and dimension n, we have that

$$\vartheta_{\infty,\min} = \lim_{r \to 0} \inf_{x \in X} \frac{\mathcal{H}^N(\mathbf{B}_g(x,\tau))}{v(n, \kappa/(n-1), r)} > 0.$$

Isoperimetric clusters and multi-isoperimetric problem

An **isoperimetric cluster of volume v** $\in \mathbb{R}_+^N$ is an N-cluster $\mathcal{E}$ that solves the minimizing problem below which is also known as **multi-isoperimetric problem**, i.e., such that $\mathbf{v}_g(\mathcal{E}) = \mathbf{v}$ and

$$\mathcal{P}_g(\mathcal{E}) = \inf\left\{\mathcal{P}_g(\mathcal{E}') : \mathcal{E}' \text{ is an N-cluster with } \mathbf{v}_g\left(\mathcal{E}'\right) = \mathbf{v}\right\}.$$

An **isoperimetric cluster of volume** $\mathbf{v} \in \mathbb{R}_{+}^{N}$ is an N-cluster $\mathcal{E}$ that solves the minimizing problem below which is also known as **multi-isoperimetric problem**, i.e., such that $\mathbf{v}_g(\mathcal{E}) = \mathbf{v}$ and

$$\mathcal{P}_g(\mathcal{E}) = \inf \left\{ \mathcal{P}_g(\mathcal{E}') : \mathcal{E}' \text{ is an N-cluster with } \mathbf{v}_g\left(\mathcal{E}'\right) = \mathbf{v} \right\}.$$

The **multi-isoperimetric profile** is a function $\mathbf{I}_{(M,g)}$ from $(0, \mathbf{Vol}_g(M))^N$ to $(0, +\infty)$ given by

$$\mathbf{I}_{(M,g)}(\mathbf{v}) = \inf \left\{ \mathcal{P}_g(\mathcal{E}) : \mathcal{E} \text{ is an N-cluster in } (M^n, g) \text{ with } \mathbf{v}_g(\mathcal{E}) = \mathbf{v} \right\}.$$

Multi-bubble conjecture

M. Hutchings, F. Morgan, M. Ritoré, and A. Ros, 2002

The double bubble conjecture holds true in $\mathbb{R}^3$.

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The double bubble conjecture holds in $\mathbb{R}^n$.

E. Milman and J. Neeman, 2022

The multi-bubble conjecture holds in $\mathbb{R}^n$ and $\mathbb{S}^n$ for all combinations of N and n such that $2 \leq N + 1 \leq \min(5, n + 1)$. Namely:

1. $N = 2$, then it holds for $n \geq 2$,
2. $N = 3$, then it holds for $n \geq 3$,
3. $N = 4$, then it holds for $n \geq 4$.

Contents

About existence and compactness

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- In fact, there are examples of manifolds which does not contain isoperimetric regions. For instance, the hyperbolic paraboloid $\mathcal{Z}$ has strictly negative Ricci curvature and does not contain any isoperimetric region. In fact, $I_{\mathcal{Z}} = I_{\mathbb{R}^2}$.

About existence and compactness

Existence of isoperimetric clusters and compactness of sequences of finite perimeter sets is a subtle point in the theory of general Riemannian manifolds.

- In fact, there are examples of manifolds which does not contain isoperimetric regions. For instance, the hyperbolic paraboloid $\mathcal{Z}$ has strictly negative Ricci curvature and does not contain any isoperimetric region. In fact, $I_{\mathcal{Z}} = I_{\mathbb{R}^2}$.
- We do not have a characterization of manifolds that contains its isoperimetric sets.

A first compactness result

- We now fix $\alpha_{ij} = 1$ for $i \neq j$ and $\alpha_{ii} = 0$.

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Proposition - *Maggi, 2012*

If $\{\mathcal{E}_k\}_{k\in\mathbb{N}}$ is a sequence of N-clusters in (M^n, g),

$$\sup_{k\in\mathbb{N}} \mathcal{P}_g\left(\mathcal{E}_k\right) < +\infty,$$

$$\inf_{k\in\mathbb{N}} \min_{1\leq h\leq N} \mathbf{Vol}_g\left(\mathcal{E}_k(h)\right) > 0$$

and

$$\mathcal{E}_k(h) \subset \mathbf{B}_g(p, R)\,, \ \ \forall k \in \mathbb{N}, h = 1, \dots, N,$$

$R > 0$, for some $p \in M$, then there exist an N-cluster $\mathcal{E}$ in (M^n, g) with $\mathcal{E}(h) \subset \mathbf{B}_g(p, R)$ such that up to a subsequence $\mathcal{E}_k \to \mathcal{E}$ as $k \longrightarrow +\infty$.

Bounded geometry and C^0-bounded geometry

Definition

We say that a smooth Riemannian manifold (M^n, g) has **bounded geometry** if there exists a constant $k \in \mathbb{R}$, such that $Ric_g \geq k(n-1)$ (i.e., $Ric_g \geq k(n-1)g$ in the sense of quadratic forms) and $\mathbf{Vol}_g\left(\mathbf{B}_M(p, inj_M)\right) \geq v_0$ for some positive constant v_0, where $\mathbf{B}_M(p, r)$ is the geodesic ball of M centered at p and of radius $r \in (0, inj_M)$.

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Definition

We say that a smooth Riemannian manifold (M^n, g) has C^0**-bounded geometry** if it has bounded geometry and satisfies:

- for every diverging sequence of points (p_j), there exist a subsequence (p_{j_l}) and a pointed smooth manifold $(M_\infty, g_\infty, p_\infty)$ with g_∞ of class C^0 such that the sequence of pointed manifolds $(M, g, p_{j_l}) \to (M_\infty, g_\infty, p_\infty)$, in C^0-topology.

Generalized compactness for sequence of N-cluster

Theorem - *R., 2019*

Suppose that (M^n, g) has C^0-bounded geometry. Let $\{\mathcal{E}_k\}_{k\in\mathbb{N}}$ be a sequence of N-clusters in (M^n, g) with $\mathcal{P}_g(\mathcal{E}_k) \leq P$ and $\mathbf{v}_g(\mathcal{E}_k)(h) \leq \mathbf{v}(h)$, for $h \in \{1, ..., N\}$. Then, up to a subsequence, there exists $J \in \mathbb{N} \cup \{+\infty\}$ such that, for all $j \in \{1, ...J\}$, there exist a sequence of points $(p^h_{jk})_{k\in\mathbb{N}} \subset M$, a manifold $(M_\infty(h), g_\infty)$, $(p^h_{j\infty})_{k\in\mathbb{N}} \subset M_\infty(h)$ and a finite perimeter set $\mathcal{E}_\infty(h) \subset M_\infty(h)$, $1 \leq h \leq N$, such that

$$(\mathcal{E}_k(h), g, p^h_{jk}) \text{ converges to } (\mathcal{E}_\infty(h), g_\infty, p^h_{j\infty})$$

in the multipointed C^0-topology. Moreover, if we define the N-cluster $\mathcal{E}_\infty = \{\mathcal{E}_\infty(h)\}_{h=1}^N$ in the manifold $M \cup \left(\cup_{h=1}^N M_\infty(h)\right)$, then $\mathbf{v}_{g_\infty}(E_\infty) = \lim_{k\to+\infty} \mathbf{v}_g(\mathcal{E}_k)$ and $\mathcal{P}_{g_\infty}(\mathcal{E}_\infty) = \lim_{k\to+\infty} \mathcal{P}_g(\mathcal{E}_k)$.

Generalized existence of isoperimetric cluster

Theorem - *R., 2019*

Suppose that (M^n, g) has C^0-bounded goemetry. Let $\{\mathcal{E}_k\}_{k\in\mathbb{N}}$ be a minimizing sequence of N-clusters for $\mathbf{v} \in \mathbb{R}^N_+$. Then, up to a subsequence, there exists $J \in \mathbb{N}$, a manifold (M_∞, g_∞), J sequences of points $(p^h_{jk})_{k\in\mathbb{N}} \subset M$, $(p^h_{j\infty})_{k\in\mathbb{N}} \subset M_\infty$ and a N-cluster $\mathcal{E}_\infty$ in (M_∞, g_∞) such that

$$(\mathcal{E}_k(h), g, p^h_{jk}) \text{ converges to } (\mathcal{E}_\infty(h), g_\infty, p^h_{j\infty}),$$

for $h \in \{1, ..., N\}$, in the multipointed C^0-topology. Moreover, $\mathbf{v}_{g_\infty}(E_\infty) = \mathbf{v}$ and $\mathcal{P}_{g_\infty}(\mathcal{E}_\infty) = \mathbf{I}_{(M_\infty, g_\infty)}(\mathbf{v}) = \mathbf{I}_{(M,g)}(\mathbf{v})$.

"Classical" existence as an application

Definition

We say that (M^n, g) is C^0-locally asymptotically a space form, if it has C^0-bounded geometry and for every diverging sequence of points (p_k) we have

$$(M, g, p_k) \to (\mathbb{M}^n_\kappa, g_{\text{standard}}, x)$$

in the C^0-topology, where $\mathbb{M}^n_\kappa$ is a n-dimensional space form of curvature κ and x is any point in $\mathbb{M}^n_\kappa$.

"Classical" existence as an application

For this special kind of manifolds, we do have the existence of isoperimetric cluster in M itself.

Theorem - *R., 2019*

Let (M^n, g) be C^0-locally asymptotically the n-dimensional space form $\mathbb{M}^n_k$ of curvature k, $Ric_g \geq k(n-1)$. Then, for every $\mathbf{v} \in \mathbb{R}^N_+$, there exist an isoperimetric cluster, i.e. an N-cluster $\mathcal{E}$ with

$$\mathbf{I}_{(M,g)}(\mathbf{v}) = \mathcal{P}_g\left(\mathcal{E}\right).$$

Boundedness of isoperimetric clusters

- F. Morgan proved the boundedness of isoperimetric cluster in Euclidean spaces.

Theorem - *Morgan's book*

Let (M^n, g) be a Riemannian manifold with bounded geometry, then isoperimetric clusters are bounded.

Contents

Theorem - *S. Nardulli and L. E. O. Acevedo, 2018*

Let (M^n, g) be a complete Riemannian manifold with bounded bounded geometry satisfying, for some positive constant $\lambda > 0$, that

$$\lim_{v \to 0^+} \frac{I(v)}{v^{\frac{n-1}{n}}} = \lambda.$$

Then there exist two positive constants $\mu^* = \mu^*(n, \kappa, inj_M, \lambda) > 0$ and $v^* = v^*(n, \kappa, inj_M, \lambda) > 0$ such that whenever $\Omega \subseteq M$ is an isoperimetric region of volume $0 \leq v \leq v^*$ it holds that

$$\operatorname{diam}_g(\Omega) \leq \mu^* v^{\frac{1}{n}}.$$

Isoperimetric sets in RCD

- We say that $(X, \mathrm{d}, \mathcal{H}^n)$ is a ncRCD(κ, n), if $(X, \mathrm{d}, \mathcal{H}^n)$ and $\mathcal{H}^n(\mathbf{B}_g(x,1)) \geq v_0 > 0$ for any $x \in X$.

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Theorem - *G. Antonelli, E. Pasqualetto, M. Pozzetta, and D. Semola, 2022*

There exist constants $\bar{v} = \bar{v}\,(\kappa, n, v_0) > 0$ and $C = C\,(K, N, v_0) > 0$ such that the following holds. Let $(X,\ \mathrm{d}, \mathcal{H}^n)$ be an ncRCD(κ, n) space. Let $E \subseteq X$ be an isoperimetric region. Then

$$\operatorname{diam} E \leq C\mathcal{H}^n(E)^{\frac{1}{n}} \quad \text{whenever } \mathcal{H}^n(E) \leq \bar{v}.$$

Theorem - *G. Antonelli, S. Nardulli, and M. Pozzetta, 2022*

Let $(X_i, \mathrm{d}_i, \mathcal{H}^n)$ be a sequence of ncRCD(κ, n) spaces, and let $E_i \subset X_i$ be bounded sets of finite perimeter such that $\sup_i (P(E_i) + \mathcal{H}^n(E_i)) < +\infty$. Then, up to subsequence, there exist a nondecreasing, possibly unboundend, sequence $\{J_i\}_{i \in \mathbb{N}} \subseteq \mathbb{N}$, points $p_{i,j} \in X_i$, with $1 \leq j \leq J_i$ for any i, and pairwise disjoint subsets $E_{i,j} \subset E_i$ such that

- $\lim_i \mathrm{d}_i(p_{i,j}, p_{i,\ell}) = +\infty$, for any $j \neq \ell < \bar{J} + 1$, where $\bar{J} := \lim_i J_i \in \mathbb{N} \cup \{+\infty\}$;
- For every $1 \leq j < \bar{J} + 1$, the sequence $(X_i, \mathrm{d}_i, \mathcal{H}^n, p_{i,j})$ converges in the pmGH sense to a pointed RCD(κ, n) space $\left(Y_j, \ \mathrm{d}_{Y_j}, \mathcal{H}^n, p_j\right)$ as $i \to +\infty$;
- there exist sets $F_j \subset Y_j$ such that $E_{i,j} \to_i F_j$ in L^1-strong and there holds

$$\lim_i \mathcal{H}^n(E_i) = \sum_{j=1}^{\bar{J}} \mathcal{H}^n(F_j), \quad \sum_{j=1}^{\bar{J}} P(F_j) \leq \liminf_i P(E_i).$$

Theorem - *G. Antonelli, S. Nardulli, and M. Pozzetta, 2022*

Moreover, if E_i is an isoperimetric set in X_i for any i, then F_j is an isoperimetric set in Y_j for any $j < \bar{J} + 1$ and

$$P(F_j) = \lim_i P(E_{i,j}),$$

for any $j < \bar{J} + 1$.

L^1-strong: Let $\{(X_i, d_i, m_i, x_i)\}_{i \in \mathbb{N}}$ be a sequence of pointed metric measure spaces converging in the pmGH sense to a pointed metric measure space (Y, d_Y, μ, y) and let (Z, d_Z) be a complete separable metric space where every (X_i, d_i) and (Y, d_Y) can be isometrically embedded. We say that a sequence of Borel sets $E_i \subset X_i$ such that $m_i(E_i) < +\infty$ for any $i \in \mathbb{N}$ converges in the L^1-strong sense to a Borel set $F \subset Y$ with $\mu(F) < +\infty$ if $m_i(E_i) \to \mu(F)$ and $\chi_{E_i} m_i \to \chi_F \mu$ with respect to the duality with continuous bounded functions with bounded support on Z.

Theorem - *Nardulli and R., 2022*

Let $(X_i, \mathrm{d}_i, \mathcal{H}^n)$ be a sequence of ncRCD(κ, n) spaces, and let $\mathcal{E}_i \subset X_i$ be bounded N-clusters such that $\sup_i \left(\mathcal{P}_g(\mathcal{E}_i) + \sum_{h=1}^N \mathbf{v}(\mathcal{E}_i)(h)\right) < +\infty$. Then, up to subsequence, there exist a nondecreasing, possibly unboundend, sequence $\{J_i\}_{i\in\mathbb{N}} \subseteq \mathbb{N}$, points $p_{i,j} \in X_i$, with $1 \le j \le J_i$ for any i, and pairwise disjoint subclusters $\mathcal{E}_{i,j}$ such that $\mathcal{E}_{i,j}(h) \subset \mathcal{E}_i(h), \forall h \in \{1, \cdots N\}$, such that

- $\lim_i \mathrm{d}_i(p_{i,j}, p_{i,\ell}) = +\infty$, for any $j \neq \ell < \bar{J} + 1$, where $\bar{J} := \lim_i J_i \in \mathbb{N} \cup \{+\infty\}$;
- For every $1 \le j < \bar{J} + 1$, the sequence $(X_i, \mathrm{d}_i, \mathcal{H}^n, p_{i,j})$ converges in the pmGH sense to a pointed RCD(κ, n) space $\left(Y_j,\ \mathrm{d}_{Y_j}, \mathcal{H}^n, p_j\right)$ as $i \to +\infty$;
- there exist clusters $\mathcal{F}_j$ in Y_j such that $\mathcal{E}_{i,j} \to_i \mathcal{F}_j$ in L^1-strong and there holds

$$\lim_i \mathbf{v}_g(\mathcal{E}_i) = \sum_{j=1}^{\bar{J}} \mathbf{v}_g(\mathcal{F}_j), \quad \sum_{j=1}^{\bar{J}} \mathcal{P}_g(\mathcal{F}_j) \le \liminf_i \mathcal{P}_g(\mathcal{E}_i).$$

Theorem - *Nardulli and R., 2022*

Moreover, if $\mathcal{E}_i$ is an isoperimetric set in X_i for any i, then $\mathcal{F}_j$ is an isoperimetric set in Y_j for any $j < \bar{J} + 1$ and

$$\mathcal{P}_g\left(\mathcal{F}_j\right) = \lim_i \mathcal{P}_g\left(\mathcal{E}_{i,j}\right),$$

for any $j < \bar{J} + 1$.

Small volumes implies small diameters

Theorem - *Nardulli and R., 2022*

Let (M^n, g) be a closed Riemannian manifold and $N = 2$, i.e. the double bubble case. There exist two constants $\mu^* = \mu^*(M, g), \mathrm{v}^* = \mathrm{v}^*(M, g) > 0$ such that for any isoperimetric cluster $\mathcal{E}$ satisfying $\mathrm{v}_g(\mathcal{E}) \in (0, \mathrm{v}^*]$, it follows

$$\mathrm{diam}_g(\mathcal{E}(1) \cup \mathcal{E}(2)) \leqslant \mu^* \left(\sum_{h=1}^{N} \mathrm{v}_g(\mathcal{E}(h)) \right)^{1/n}.$$

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Proof: We apply the last theorem to

$$(X_i, \mathrm{d}_i, \mathcal{H}_i^n) := \left(M^n, \mathrm{v}_i^{-\frac{1}{n}} \mathrm{d}_g, \mathcal{H}_{g_i}^n \right), \forall i \in \mathbb{N}.$$

Proof

We obtain that $(Y_j^\infty, \mathrm{d}_{Y_j^\infty}) = (\mathbb{R}^n, g_{euc})$ for every $j \in [1, \bar{J}+1) \cap \mathbb{N}$ and the existence of a isoperimetric 2-cluster in $\mathbb{R}^n$ as follows

$$\mathcal{E}^\infty := \left(\cup_{j=1}^{\bar{J}} \mathcal{E}_k^\infty(1), \cup_{j=1}^{\bar{J}} \mathcal{E}_k^\infty(2) \right).$$

So, we proceed with the following computations

$$\begin{aligned}
\mathcal{P}_{g_{euc}}\left(\mathcal{E}^\infty\right) &\leq \liminf_{k \to +\infty} \mathcal{P}_{g_k}\left(\mathcal{E}_k\right) = \liminf_{k \to +\infty} \frac{\mathcal{P}_g\left(\mathcal{E}_k\right)}{\mathrm{v}_k^{\frac{n-1}{n}}} \\
&= \liminf_{k \to +\infty} \frac{\mathbf{I}_{(M,g)}\left(\mathbf{v}_g\left(\mathcal{E}_k\right)\right)}{\mathrm{v}_k^{\frac{n-1}{n}}} \\
&\leq \liminf_{k \to +\infty} \frac{\mathbf{I}_{(\mathbb{R}^n, g_{euc})}\left(\mathbf{v}_g\left(\mathcal{E}_k\right)\right)}{\mathrm{v}_k^{\frac{n-1}{n}}} \\
&= \liminf_{k \to +\infty} \mathbf{I}_{(\mathbb{R}^n, g_{euc})}\left(\frac{\mathbf{v}_g\left(\mathcal{E}_k\right)}{\mathrm{v}_k}\right) = \mathbf{I}_{(\mathbb{R}^n, g_{euc})}\left(\lambda, \mu\right),
\end{aligned}$$

where, up to a subsequence,

$$\lambda = \lim_{k\to+\infty} \frac{\mathrm{v}_g\left(\mathcal{E}_k(1)\right)}{\mathrm{v}_k} \text{ and } \mu = \lim_{k\to+\infty} \frac{\mathrm{v}_g\left(\mathcal{E}_k(2)\right)}{\mathrm{v}_k}.$$

We assume by contradiction that $\bar{J} > 1$, then we have that $\mathcal{E}^\infty$ is an isoperimetric 2-cluster in $\mathbb{R}^n$ such that $\mathcal{E}^\infty(1) \cup \mathcal{E}^\infty(2)$ is a disconnected set. So, it is clearly a contradiction with either the double bubble conjecture, if $\lambda, \mu > 0$, or the classical solution of the Euclidean isoperimetric problem, if $\lambda = 0$ or $\mu = 0$. Therefore, $\bar{J} = 1$ which is solvable using the technique called 'selecting a large subdomain'.

Small volumes implies small diameters

Conjecture (working in progress - Nardulli and R.)

Let (M^n, g) be a closed Riemannian manifold and $N = 2$, i.e. the double bubble case. There exist two constants $\mu^* = \mu^*(n, N, \kappa, v_0), \mathrm{v}^* = \mathrm{v}^*(n, N, \kappa, v_0) > 0$ such that for any isoperimetric cluster $\mathcal{E}$ satisfying $\mathrm{v}_g(\mathcal{E}) \in (0, \mathrm{v}^*]$, it follows

$$\mathrm{diam}_g(\mathcal{E}(1) \cup \mathcal{E}(2)) \leqslant \mu^* \left(\sum_{h=1}^{N} \mathrm{v}_g(\mathcal{E}(h)) \right)^{1/n}.$$

Contents

Local Holder continuity of the multi-isoperimetric profile

Theorem

Let (M^n, g) be a manifold with bounded geometry. Then there exists a constant $C(n, N, \kappa, v_0) > 0$ such that for every $\mathbf{v}, \mathbf{v}' \in]0, \mathbf{Vol}_g(M)[^N$ satisfying $\mathbf{v}' \in \mathbf{B}_{\mathbb{R}^N}(\mathbf{v}, R_{\mathbf{v}})$, where

$$R_{\mathbf{v}} = \frac{1}{C(n,N,k)} \min\left\{ v_0, \sum_{h=1}^{N} \left(\frac{\mathbf{v}(h)}{I_M(\mathbf{v}) + C(n,k)} \right)^n \right\},$$

we have that

$$|\mathbf{I}_{(M,g)}(\mathbf{v}) - \mathbf{I}_{(M,g)}(\mathbf{v}')| \leq C(n,k) \left(\frac{|\mathbf{v} - \mathbf{v}'|}{v_0} \right)^{\frac{n-1}{n}}.$$

Contents

Further problems

- Generalize the results for the nonsmooth case, i.e., consider a RCD space instead of smooth manifolds (M^n, g).
- Extend the results for the weighted perimeter of clusters.

- Resende de Oliveira, R. (2022). *On Clusters and the Multi-isoperimetric Profile in Riemannian Manifolds with Bounded Geometry*. Journal of Dynamical and Control Systems.
- J.H. Andrade, J. Conrado, S. Nardulli, P. Piccione, and R. Resende (2022). *Multiplicity of solutions to the multiphasic Allen–Cahn–Hilliard system with a small volume constraint on closed parallelizable manifolds*. Forthcoming revised version.

Thank you for your attention!