Lecture 3. Topological properties of isoperimetric sets in RCD spaces

Isoperimetric Problems

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$\Longrightarrow$ **Sobolev spaces** on metric measure spaces were thoroughly studied.

Hajłasz, Cheeger, Shanmugalingam, Ambrosio, Gigli, Savaré, Di Marino...

A way to define $W^{1,2}(\mathrm{X})$, with $(\mathrm{X}, \mathsf{d}, \mathfrak{m})$ metric measure space: as the finiteness domain of the **Cheeger energy** $\mathrm{Ch}\colon L^2(\mathrm{X}) \to [0, +\infty]$, which is the $L^2(\mathrm{X})$-lower semicontinuous envelope of the functional

$$L^2(\mathrm{X}) \ni f \longmapsto \begin{cases} \frac{1}{2}\int \mathrm{lip}^2(f)\,\mathrm{d}\mathfrak{m}, & \text{if } f \in \mathrm{LIP}_{bs}(\mathrm{X}), \\ +\infty, & \text{otherwise.} \end{cases}$$

Here, $\mathrm{lip}(f)$ stands for the **slope**: $\mathrm{lip}(f)(x) := \overline{\lim}_{y \to x} \frac{|f(y)-f(x)|}{\mathsf{d}(y,x)}$.

$\Longrightarrow$ The Cheeger energy admits the following integral representation:

$$\mathrm{Ch}(f) = \frac{1}{2}\int |Df|^2\,\mathrm{d}\mathfrak{m}, \quad \text{for every } f \in W^{1,2}(\mathrm{X}).$$

The function $|Df| \in L^2(\mathrm{X})$ is called the **minimal relaxed slope** of f.

Sobolev and vector calculus on metric measure spaces

When Ch is a quadratic form, we say that $(\mathrm{X}, \mathsf{d}, \mathfrak{m})$ is **infinitesimally Hilbertian**. In this case, the *carré du champ* operator is bilinear:

$$W^{1,2}(\mathrm{X}) \times W^{1,2}(\mathrm{X}) \ni (f, g) \mapsto \nabla f \cdot \nabla g := \frac{|D(f+g)|^2 - |Df|^2 - |Dg|^2}{2}.$$

A notion of *gradient* was introduced by [Gigli'18]: the relevant object is the **tangent module** $L^2(T\mathrm{X})$, i.e. the completion of $L^\infty(\mathrm{X})$-linear combinations of the 'formal' gradients ∇f of $f \in W^{1,2}(\mathrm{X})$.

DIVERGENCE: $v \in L^2(T\mathrm{X})$ has **divergence** $\mathrm{div}(v) \in L^2(\mathrm{X})$ if

$$\int \nabla f \cdot v \, \mathrm{d}\mathfrak{m} = -\int f \, \mathrm{div}(v) \, \mathrm{d}\mathfrak{m}, \qquad \text{for every } \in W^{1,2}(\mathrm{X}).$$

LAPLACIAN: $f \in W^{1,2}(\mathrm{X})$ has **Laplacian** Δf if $\exists \, \mathrm{div}(\nabla f) =: \Delta f$.

$\Longrightarrow$ The **heat flow** semigroup $(\mathsf{h}_t)_{t \geq 0}$, i.e. the gradient flow of the Cheeger energy, is characterised by the identity $\frac{\mathrm{d}}{\mathrm{d}t} \mathsf{h}_t f = \Delta \mathsf{h}_t f$.

The class of RCD spaces

$\Longrightarrow$ **RCD spaces** are 'Riemannian-like' metric measure spaces verifying *lower bounds on the Ricci curvature* and *upper bounds on the dimension*.

Lott, Villani, Sturm, Bacher, Ambrosio, Gigli, Savaré, Rajala, Mondino, Erbar, Kuwada, Cavalletti, Milman...

A metric measure space $(\mathrm{X}, \mathsf{d}, \mathfrak{m})$ is called an **RCD(K, N) space**, for some constants $K \in \mathbb{R}$ and $N \in [1, \infty)$, provided the following hold:

i) There exist $C > 0$ and $\bar{x} \in \mathrm{X}$ such that $\mathfrak{m}(B_r(\bar{x})) \leq Ce^{Cr^2}$ $\forall r > 0$.

ii) **(Sobolev-to-Lipschitz)** Each function $f \in W^{1,2}(\mathrm{X})$ with $|Df| \leq 1$ $\mathfrak{m}$-a.e. has a 1-Lipschitz representative.

iii) $(\mathrm{X}, \mathsf{d}, \mathfrak{m})$ is infinitesimally Hilbertian.

iv) **(Weak Bochner inequality)** For sufficiently many functions f,

$$\Delta \frac{|Df|^2}{2} \geq \frac{(\Delta f)^2}{N} + \nabla f \cdot \nabla \Delta f + K|Df|^2.$$

The class of RCD spaces

$\Longrightarrow$ $\mathrm{RCD}(K,N)$ spaces with $N \in \mathbb{N}$ and $\mathfrak{m} = \mathscr{H}^N$ have a special role.
(These are *non-collapsed*, in the sense of [De Philippis-Gigli'18].)

The theory of RCD covers the following important classes of spaces:

- Smooth **Riemannian manifolds** with Ricci curvature bounded from below (possibly *weighted* and/or *with convex boundary*).
- Finite-dimensional **Alexandrov spaces**, with sectional curvature bounded from below [Petrunin'11].
- **Ricci limits**, i.e. *limits* of sequences of Riemannian manifolds with uniform lower bounds on the Ricci curvature and having constant dimension (Cheeger, Colding, Naber...).

Limits are with respect to the **pointed measured Gromov–Hausdorff** topology (**pmGH** for short), which we will recall in the following slide.

Pointed measured Gromov–Hausdorff convergence

$\Longrightarrow$ We recall the *extrinsic definition* of **pmGH convergence**.

Let $(\mathrm{X}_n, \mathsf{d}_n, \mathfrak{m}_n, \bar{x}_n)$ be a pointed $\mathsf{RCD}(K_n, N_n)$ space, with $(K_n)_n, (N_n)_n$ bounded, and $(\mathrm{X}_\infty, \mathsf{d}_\infty, \mathfrak{m}_\infty, \bar{x}_\infty)$ a pointed metric measure space. Then

$$(\mathrm{X}_n, \mathsf{d}_n, \mathfrak{m}_n, \bar{x}_n) \overset{pmGH}{\longrightarrow} (\mathrm{X}_\infty, \mathsf{d}_\infty, \mathfrak{m}_\infty, \bar{x}_\infty)$$

if there exist a proper metric space $(\mathrm{Z}, \mathsf{d}_\mathrm{Z})$ and *isometric embeddings* $\iota_n \colon \mathrm{X}_n \hookrightarrow \mathrm{Z}$ for every $n \in \mathbb{N} \cup \{\infty\}$ such that $\iota_n(\bar{x}_n) \to \iota_\infty(\bar{x}_\infty)$ and

$$(\iota_n)_\# \mathfrak{m}_n \rightharpoonup (\iota_\infty)_\# \mathfrak{m}_\infty, \qquad \text{in duality with } C_{bs}(\mathrm{Z}).$$

FUNDAMENTAL PROPERTIES OF RCD SPACES RELATED TO PMGH:

- **(Stability)** If we assume that $K_n \to K$ and $N_n \to N$, then the pmGH-limit space $(\mathrm{X}_\infty, \mathsf{d}_\infty, \mathfrak{m}_\infty)$ is $\mathsf{RCD}(K, N)$.
- **(Compactness)** The class of $\mathsf{RCD}(K, N)$ spaces is *pmGH-compact*.

Structure theory of RCD spaces

For a pointed $\mathrm{RCD}(K, N)$ space $(\mathrm{X}, \mathsf{d}, \mathfrak{m}, x)$, the **tangent cone** $\mathrm{Tan}_x(\mathrm{X})$ is the set of all those pointed $\mathrm{RCD}(0, N)$ spaces $(\mathrm{Y}, \mathsf{d}_\mathrm{Y}, \mathfrak{m}_\mathrm{Y}, \bar{y})$ such that

$$(\mathrm{X}, r_i^{-1}\mathsf{d}, c_{x,r_i}\mathfrak{m}, x) \overset{pmGH}{\longrightarrow} (\mathrm{Y}, \mathsf{d}_\mathrm{Y}, \mathfrak{m}_\mathrm{Y}, \bar{y}), \qquad \text{as } i \to \infty,$$

for some sequence of radii $r_i \searrow 0$, where c_{x,r_i} are *normalising factors*.

$\Longrightarrow$ It holds that $\mathrm{Tan}_x(\mathrm{X}) \neq \varnothing$ for every $x \in \mathrm{X}$ by pmGH-compactness.

Theorem (Structure of RCD spaces)

Let $(\mathrm{X}, \mathsf{d}, \mathfrak{m})$ be an $\mathrm{RCD}(K, N)$ space. Then there exists a unique $n \in \mathbb{N}$ with $n \leq N$ such that $\mathrm{Tan}_x(\mathrm{X}) = \{(\mathbb{R}^n, 0)\}$ for $\mathfrak{m}$-a.e. $x \in \mathrm{X}$. Moreover, (X, d) is *n-rectifiable* up to $\mathfrak{m}$-null sets and $\mathfrak{m} = \theta\mathcal{H}^n$ for some $\theta\colon \mathrm{X} \to (0, +\infty)$. We call n the **essential dimension** of X.

Gigli, Mondino, Rajala, Naber, Bruè, Semola, Kell, Pasqualetto, De Philippis, Marchese, Rindler...

Refined vector calculus on RCD spaces

$\Longrightarrow$ The RCD condition entails a refined (and *second-order*) calculus.

The key observation, due to [Savaré'14] (see also [Gigli'18]), is that

$$\mathrm{Test}(\mathrm{X}) := \left\{ f \in W^{1,2}(\mathrm{X}) \cap \mathrm{LIP}_b(\mathrm{X}) \;\middle|\; \exists \Delta f \in W^{1,2}(\mathrm{X}) \right\},$$

is an algebra of functions strongly dense in $W^{1,2}(\mathrm{X})$ and satisfying

$$\nabla f \cdot \nabla g \in W^{1,2}(\mathrm{X}), \qquad \text{for every } f, g \in \mathrm{Test}(\mathrm{X}). \tag{1}$$

As a consequence, one deduces that $|Df| \in W^{1,2}(\mathrm{X})$ for all $f \in \mathrm{Test}(\mathrm{X})$.

$\Longrightarrow$ In particular, the **test vector fields**, which are the elements of

$$\mathrm{Test}(T\mathrm{X}) := \left\{ \sum_{i=1}^{n} g_i \nabla f_i \;\middle|\; (f_i)_{i=1}^n, (g_i)_{i=1}^n \subset \mathrm{Test}(\mathrm{X}) \right\} \subset L^2(T\mathrm{X}),$$

are well-defined *up to* Cap*-null sets*, in a suitable sense (see next slide).

Using test functions/vector fields and suitable integration-by-parts formulae, [Gigli'18] introduced **Hessian**, **covariant derivative**, etc...

The **Sobolev 2-capacity** on X is the outer measure Cap, given by

$$\mathrm{Cap}(E) := \inf_f \int |f|^2 \,\mathrm{d}\mathfrak{m} + \int |Df|^2 \,\mathrm{d}\mathfrak{m}, \qquad \text{for every set } E \subset \mathrm{X},$$

where the infimum is among all $f \in W^{1,2}(\mathrm{X})$ satisfying $f \geq 1$ $\mathfrak{m}$-a.e. on an open neighbourhood of E. In great generality, it holds that

every $f \in W^{1,2}(\mathrm{X})$ has a **quasi-continuous** representative.

In particular, Sobolev functions are well-defined Cap-almost everywhere.

By building on top of (1), in [Debin-Gigli-Pasqualetto'21] the concept of the **capacitary tangent module** $L^\infty_{\mathrm{Cap}}(T\mathrm{X})$ on $(\mathrm{X}, \mathsf{d}, \mathfrak{m})$ was introduced.

$\Longrightarrow$ $L^\infty_{\mathrm{Cap}}(T\mathrm{X})$ is obtained as the completion of the $L^\infty(\mathrm{Cap})$-linear combinations of the 'formal' gradients ∇f of $f \in \mathrm{Test}(\mathrm{X})$.

As $\mathfrak{m} \ll \mathrm{Cap}$, there is a natural projection map $L^\infty_{\mathrm{Cap}}(T\mathrm{X}) \to L^\infty(T\mathrm{X})$.

Sets of finite perimeter and functions of bounded variation

$\Longrightarrow$ **BV functions** on metric measure spaces were thoroughly studied.

Miranda Jr., Ambrosio, Di Marino, Martio...

Given any $f \in L^1_{loc}(\mathrm{X})$ and $\Omega \subset \mathrm{X}$ open, we define

$$|Df|(\Omega) := \inf \left\{ \varliminf_{n\to\infty} \int_\Omega \mathrm{lip}(f_n)\,\mathrm{d}\mathfrak{m} \;\middle|\; (f_n)_n \subset \mathrm{LIP}_{loc}(\Omega),\ f_n \to f \text{ in } L^1_{loc}(\Omega) \right\}.$$

If $|Df|(\mathrm{X}) < +\infty$, then $|Df|$ can be extended to a Borel measure $|Df|$.

- We say that $f \in L^1(\mathrm{X})$ is of **bounded variation**, briefly $f \in \mathrm{BV}(\mathrm{X})$, if $|Df|(\mathrm{X}) < +\infty$. We call $|Df|$ the **total variation measure** of f.
- $E \subset \mathrm{X}$ Borel is of **finite perimeter** if $P(E) := |D\mathbb{1}_E|(\mathrm{X}) < +\infty$. We call $P(E,\cdot) := |D\mathbb{1}_E|$ the **perimeter measure** of E.

The total variation enjoys the following *lower semicontinuity* property:

$$|Df|(\Omega) \le \varliminf_{n\to\infty} |Df_n|(\Omega), \quad \text{if } f_n \to f \text{ in } L^1_{loc}(\mathrm{X}) \text{ and } \Omega \subset \mathrm{X} \text{ is open.}$$

De Giorgi's Theorem for sets of finite perimeter in RCD

$\Longrightarrow$ De Giorgi's *Structure Theorem* for sets of finite perimeter in the Euclidean space was generalised to the setting of RCD spaces.

Following [Ambrosio-Bruè-Semola'19], given $E \subset \mathrm{X}$ of finite perimeter, we call $\mathrm{Tan}_x(\mathrm{X}, E)$ the set of $(\mathrm{Y}, \mathsf{d}_\mathrm{Y}, \mathfrak{m}_\mathrm{Y}, \bar{y}, F)$ with $(\mathrm{Y}, \bar{y}) \in \mathrm{Tan}_x(\mathrm{X})$ such that $F \subset \mathrm{Y}$ has (locally) finite perimeter and $\chi_E^{(r_i)} \to \chi_F$ in L^1_{loc} along some realisation of the pmGH-convergence $r_i^{-1}\mathrm{X} \to \mathrm{Y}$.

Theorem (Structure of sets of finite perimeter)

Let $(\mathrm{X}, \mathsf{d}, \mathfrak{m})$ be an RCD(K, N) space of essential dimension $n \leq N$ and $E \subset \mathrm{X}$ a set of finite perimeter. Then the **reduced boundary** of E,

$$\mathcal{F}E := \left\{ x \in \mathrm{X} \;\middle|\; \mathrm{Tan}_x(\mathrm{X}, E) = \left\{ (\mathbb{R}^n, 0, \{x_n > 0\}) \right\} \right\},$$

satisfies $P(E, \mathrm{X} \setminus \mathcal{F}E) = 0$. Moreover, $\mathcal{F}E$ is $(n-1)$-rectifiable up to $P(E, \cdot)$-null sets and $P(E, \cdot) = \Theta \mathcal{H}^{n-1}$, where $\Theta(x) := \lim_{r \searrow 0} \frac{\mathfrak{m}(B_r(x))}{\omega_n r^n}$.

Ambrosio, Bruè, Semola, Pasqualetto, Antonelli, Brena...

Gauss–Green formula on RCD spaces

As proved in [Bruè-Pasqualetto-Semola'22] (see also [Brena-Gigli'22]),

$$P(E,\cdot) \ll \mathrm{Cap}, \qquad \text{for every set } E \subset \mathrm{X} \text{ of finite perimeter.}$$

Therefore, the statement of the following result is meaningful:

Theorem (Gauss-Green formula)

Let $(\mathrm{X}, \mathsf{d}, \mathfrak{m})$ be an $\mathsf{RCD}(K, N)$ space, $E \subset \mathrm{X}$ a set of finite perimeter. Then there exists an element $\nu_E \in L^\infty_{\mathrm{Cap}}(T\mathrm{X})$, unique up to $P(E,\cdot)$-a.e. equality, such that $|\nu_E| = 1$ holds $P(E,\cdot)$-a.e. and

$$\int_E \mathrm{div}(v)\,\mathrm{d}\mathfrak{m} = \int v \cdot \nu_E \,\mathrm{d}P(E,\cdot), \qquad \text{for every } v \in \mathrm{Test}(T\mathrm{X}).$$

We say that ν_E is the **outer unit normal** of E.

$\Longrightarrow$ A Gauss–Green formula for vector fields having *measure-valued divergence* was obtained in [Buffa-Comi-Miranda Jr.'21].

Topological properties of isoperimetric sets in RCD

Let $(\mathrm{X}, \mathsf{d}, \mathfrak{m})$ be an $\mathsf{RCD}(K, N)$ space. A set $E \subset \mathrm{X}$ of finite perimeter with $0 < \mathfrak{m}(E) < +\infty$ is said to be **isoperimetric** provided it holds

$$P(E) \leq P(F), \qquad \text{whenever } F \subset \mathrm{X} \text{ satisfies } \mathfrak{m}(F) = \mathfrak{m}(E).$$

NOTATION: $E^{(1)}$ denotes the **essential interior** of E, where we set

$$E^{(t)} := \left\{ x \in \mathrm{X} \,\middle|\, \lim_{r \searrow 0} \frac{\mathfrak{m}(E \cap B_r(x))}{\mathfrak{m}(B_r(x))} = t \right\}, \qquad \text{for every } t \in [0,1].$$

The **essential boundary** of E is defined as $\partial^e E := \mathrm{X} \setminus (E^{(1)} \cup E^{(0)})$.

$\Longrightarrow$ Note that $\mathcal{F}E \subset \partial^e E \subset \partial E$ and that $E^{(1)} = E$ up to $\mathfrak{m}$-null sets.

Theorem (Topological properties of isoperimetric sets)

Let $(\mathrm{X}, \mathsf{d}, \mathcal{H}^N)$ be $\mathsf{RCD}(K, N)$, with $N \geq 2$ and $\inf_{x \in \mathrm{X}} \mathcal{H}^N(B_1(x)) > 0$. Let $E \subset \mathrm{X}$ be an isoperimetric set. Then $E^{(1)}$ is open and bounded. Moreover, $\partial E^{(1)} = \partial^e E$ and $\partial E^{(1)}$ is $(N-1)$-Ahlfors regular in X.

Strategy of proof:

1) Build *measure-prescribing deformations* of sets with interior points.
2) Prove that isoperimetric sets have interior and exterior points.
3) Combine 1) with 2), to obtain the *Topological Regularity Theorem*.

$\Longrightarrow$ Unless otherwise specified, hereafter the results are taken from:

Antonelli-Pasqualetto-Pozzetta, *Isoperimetric sets in spaces with lower bounds on the Ricci curvature*, Nonlinear Analysis 220 (2022), 112839.

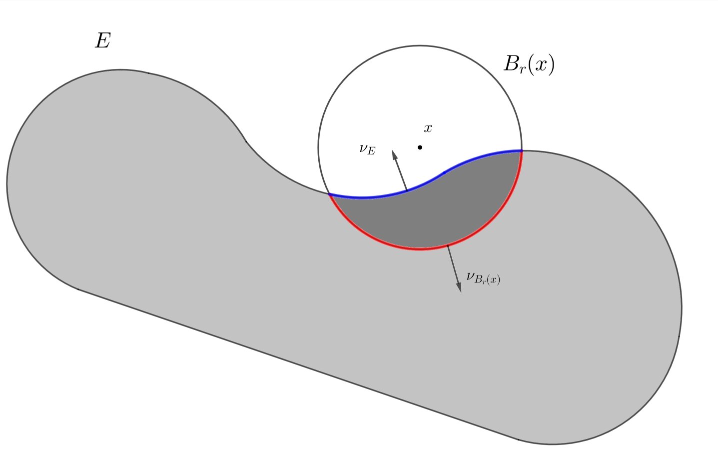
Theorem (Deformation Lemma)

Let $(\mathrm{X}, \mathrm{d}, \mathfrak{m})$ be an RCD(K, N) space and $R > 0$. Then for any $E \subset \mathrm{X}$ of finite perimeter and any point $x \in \mathrm{X}$ it holds that

$$P(B_r(x), E^{(1)}) \leq C_{K,N,R} \frac{\mathfrak{m}(E \cap B_r(x))}{r} + P(E, B_r(x)), \quad \forall r \in (0, R).$$

Deformation Lemma in RCD spaces

$$P(B_r(x), E^{(1)}) \leq C_{K,N,R} \frac{\mathfrak{m}(E \cap B_r(x))}{r} + P(E, B_r(x))$$



Deformation Lemma in RCD spaces

Sketch of proof.

By lower semicontinuity, it suffices to prove the claim for a.e. $r \in (0, R)$, thus we can assume that $P(E \cap B_r(x)) = P(B_r(x), E^{(1)}) + P(E, B_r(x))$.

$\Longrightarrow$ Apply Gauss–Green formula to $\nabla \mathrm{d}_x^2 = 2\mathrm{d}_x \nabla \mathrm{d}_x$ on $F \coloneqq E \cap B_r(x)$.

$$\nu_F = \begin{cases} \nu_{B_r(x)} = \nabla \mathrm{d}_x, & \text{on } E^{(1)} \cap \partial^e F, \\ \nu_E, & \text{on } B_r(x) \cap \partial^e F. \end{cases} \tag{2}$$

We will need the **Laplacian comparison estimate** from [Gigli'15]:

$$\Delta \mathrm{d}_x^2 \leq 2N(\tilde{\tau}_{K,N} \circ \mathrm{d}_x)\mathfrak{m}. \tag{3}$$

By applying the Gauss–Green formula to $\nabla \mathrm{d}_x^2$ on F, we obtain that

$$\underbrace{\int_F \Delta \mathrm{d}_x^2}_{(LHS)} = \int_F \operatorname{div}(\nabla \mathrm{d}_x^2) = \underbrace{\int_{\partial^e F} \nu_F \cdot \nabla \mathrm{d}_x^2 \, \mathrm{d}P(F, \cdot)}_{(RHS)}.$$

Deformation Lemma in RCD spaces

We can bound (LHS) using the Laplacian comparison estimate for d_x^2:

$$(LHS) \overset{(3)}{\leq} 2N \int_F \tilde{\tau}_{K,N} \circ \mathsf{d}_x \, \mathrm{d}\mathfrak{m} \leq 2N \tilde{C}_{K,N,R} \, \mathfrak{m}(E \cap B_r(x)).$$

Concerning (RHS), we can estimate

$$\begin{aligned}
(RHS) &= 2 \int_{\partial^e F} \mathsf{d}_x (\nu_F \cdot \nabla \mathsf{d}_x) \, \mathrm{d}P(F, \cdot) \\
&\overset{(2)}{=} 2 \int_{E^{(1)}} \underbrace{\mathsf{d}_x |\nabla \mathsf{d}_x|^2}_{= r \text{ on } \partial B_r(x)} \, \mathrm{d}P(B_r(x), \cdot) + 2 \int_{B_r(x)} \mathsf{d}_x (\nu_E \cdot \nabla \mathsf{d}_x) \, \mathrm{d}P(E, \cdot) \\
&\geq 2r \, P(B_r(x), E^{(1)}) - 2 \int_{B_r(x)} \underbrace{\mathsf{d}_x |\nu_E \cdot \nabla \mathsf{d}_x|}_{\leq r \text{ on } B_r(x)} \, \mathrm{d}P(E, \cdot) \\
&\geq 2r \, P(B_r(x), E^{(1)}) - 2r \, P(E, B_r(x)).
\end{aligned}$$

We thus obtain the statement with $C_{K,N,R} := N \tilde{C}_{K,N,R}$. □

Volume-prescribing localised deformations

We say that a Borel set $E \subset \mathrm{X}$ has an **interior point** $x \in \mathrm{X}$ provided

$$\mathfrak{m}(B_r(x) \setminus E) = 0, \qquad \text{for some radius } r > 0.$$

Corollary (Measure-prescribing deformations)

Let $E \subset \mathrm{X}$ be of finite perimeter. Suppose E has an interior point. Then there exist $\bar{v}, \bar{C} > 0$ and a ball B such that the following holds: given any $v \in (0, \bar{v})$, there exists a Borel set $F \subset \mathrm{X}$ with $E \subset F$ and

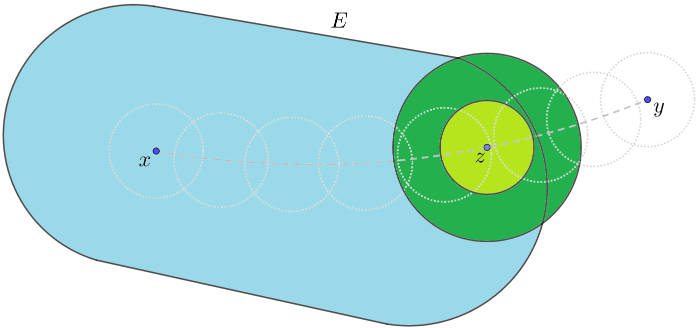
$$E \Delta F \subset B, \quad \mathfrak{m}(F \cap B) = \mathfrak{m}(E \cap B) + v, \quad P(F) \leq \bar{C} v + P(E).$$

Proof.
Let $x \in \mathrm{X}$ and $r > 0$ satisfy $\mathfrak{m}(B_r(x) \setminus E) = 0$. On a geodesic joining x with any $y \in E^{(0)}$, one can pick a point z such that $\mathfrak{m}(B_{r/2}(z) \setminus E) = 0$ and $\mathfrak{m}(B_r(z) \setminus E) > 0$. We conclude by using the Deformation Lemma: one can choose $B := B_r(z)$, $\bar{v} := \mathfrak{m}(B_r(z) \setminus E)$, and $\bar{C} := \frac{2C_{K,N,r}}{r}$. □

Volume-prescribing localised deformations

For any $v \in (0, \bar{v})$ there exists $\rho \in (\frac{r}{2}, r)$ such that $F := E \cap B_\rho(z)$ satisfies $\mathfrak{m}(F \cap B_r(z)) = \mathfrak{m}(E \cap B_r(z)) + v$. By the Deformation Lemma,

$$P(F) \leq P(E) + C_{K,N,r} \frac{\mathfrak{m}(B_\rho(z) \setminus E)}{\rho} \leq P(E) + \frac{2C_{K,N,r}}{r} v.$$



Isoperimetric sets have interior and exterior points

Hereafter, we consider an $\mathrm{RCD}(K,N)$ space $(\mathrm{X},\mathsf{d},\mathcal{H}^N)$ with $N \geq 2$ and

$$\inf_{x\in\mathrm{X}} \mathcal{H}^N(B_1(x)) > 0.$$

Some important properties of this class of spaces:

i) **(Bishop–Gromov comparison)** If $x \in \mathrm{X}$ and $0 < r < R$, then

$$\frac{\mathcal{H}^N(B_R(x))}{v(N,K/(N-1),R)} \leq \frac{\mathcal{H}^N(B_r(x))}{v(N,K/(N-1),r)},$$

where $v(N,K/(N-1),r)$ is the volume of an r-ball in $\mathbb{M}^N_{K/(N-1)}$.

ii) Let $\Theta_N(x) \coloneqq \lim_{r\searrow 0} \frac{\mathcal{H}^N(B_r(x))}{\omega_N r^N}$. Then $\Theta_N = 1$ holds $\mathcal{H}^N$-a.e. on X.

iii) By lower semicontinuity of Θ_N, we deduce from ii) that $\Theta_N \leq 1$.

Proposition

Every isoperimetric set $E \subset \mathrm{X}$ has both interior and exterior points.

Isoperimetric sets have interior and exterior points

To prove the previous Proposition, we adapted an argument by [Xia'05], which was in turn inspired by [Gonzalez-Massari-Tamanini'83].

$\Longrightarrow$ We omit the details. Some of the ingredients of the proof:

1) A **volume decay estimate**: given any $o \in E^{(0)}$, it holds that

$$\inf \left\{ \mathcal{H}^N(E \cap B_r(x)) \,\middle|\, x \in B_{\bar{r}}(o) \right\} \leq C r^{\frac{N^2}{N-1}}, \quad \forall r \in (0, \bar{r}).$$

2) **Almost Euclidean isoperimetric ineq.**: if $\Theta_N(o) = 1$ and $\varepsilon > 0$,

$$P(E) \geq N\omega_N^{1/N}(1 - \varepsilon - \bar{C}r)\mathcal{H}^N(E)^{\frac{N-1}{N}},$$

when $r < \bar{r}$, $x \in B_{\bar{R}}(o)$, $E \subset B_r(x)$. See [Cavalletti-Mondino'20].

3) Balls *almost* verify the reverse Euclidean isoperimetric inequality:

$$P(B) \leq N\omega_N^{1/N}(1 + \varepsilon)\mathcal{H}^N(B)^{\frac{N-1}{N}}, \quad \forall \, \text{ball } B \text{ with } \mathcal{H}^N(B) \leq \bar{v}_\varepsilon.$$

$\Longrightarrow$ Thanks to the RCD version of the **Morgan–Johnson Lemma**.

Topological regularity of isoperimetric sets

All in all, we have *measure-prescribing deformations* of isoperimetric.

$\Longrightarrow$ Following e.g. [Maggi'12], we prove the *Topological Regularity Thm*.

MAIN STEPS OF THE PROOF:

1) We prove that every isoperimetric set $E \subset \mathrm{X}$ is a **(Λ, r_0)-perimeter minimiser**, for some $\Lambda, r_0 > 0$. This means that if $x \in \mathrm{X}$ and $r < r_0$,

$$P(E, B_r(x)) \leq P(F, B_r(x)) + \Lambda \mathcal{H}^N(E \Delta F), \quad \text{when } E \Delta F \Subset B_r(x).$$

2) **Isoperimetric inequality for small volumes**: $\exists \bar{C}, \bar{v} > 0$ such that

$$E \subset \mathrm{X} \text{ with } \mathcal{H}^N(E) \leq \bar{v} \quad \Longrightarrow \quad P(E) \geq \bar{C} \mathcal{H}^N(E)^{\frac{N-1}{N}}.$$

3) Using 2), we show that E is also a **(K, r_0')-quasi minimal** set for some $K \geq 1$ and $r_0' > 0$. This means that if $x \in \mathrm{X}$ and $r < r_0'$,

$$P(E, B_r(x)) \leq K\, P(F, B_r(x)), \quad \text{when } E \Delta F \Subset B_r(x).$$

Topological regularity of isoperimetric sets

4) We prove that there exist $C_1 \in (0,1)$, $C_2 \geq 1$, and $\bar{r} > 0$ such that

$$C_1 \leq \frac{\mathcal{H}^N(E \cap B_r(x))}{\mathcal{H}^N(B_r(x))} \leq 1 - C_1, \quad \frac{1}{C_2} \leq \frac{P(E, B_r(x))}{r^{N-1}} \leq C_2,$$

for every $x \in \partial E^{(1)}$ and $r < \bar{r}$. It follows that $E^{(1)}$ is open, that $\partial E^{(1)} = \partial^e E$, and that the set $\partial E^{(1)}$ is $(N-1)$-Ahlfors regular.

5) To prove that $E^{(1)}$ *is bounded*: fix an interior point $\bar{x}$ of E and define $V(r) := \mathcal{H}^N(E \setminus B_r(x))$ for every $r > 0$. One can show that

$$V(r)^{\frac{N-1}{N}} \leq CV'(r), \qquad \text{for a.e. } r \text{ sufficiently large},$$

for some $C > 0$. By an *ODE comparison*, we deduce that $V(\bar{r}) = 0$ for some $\bar{r} > 0$. This means that $\mathcal{H}^n(E \setminus B_{\bar{r}}(\bar{x})) = 0$, as desired. $\square$

Thank you for the attention