Minimal clusters in the plane with double densities

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Isoperimetric Problems

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based on joint works with A. Pratelli (UNIPI), G. Stefani (SISSA)

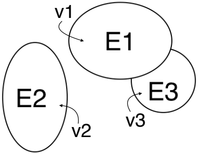
Outline

1. Intro: Isoperimetric clustering problem
2. First part: Isotropic perimeter density
3. Second part: The anisotropic case

The classical isoperimetric clustering problem

A cluster is a collection of *pairwise disjoint* sets in $\mathbb{R}^n$.

$$\mathcal{E} = \{E_1, E_2, \ldots, E_m\}, \quad m \geq 1$$



Volume: $|\mathcal{E}| = (|E_1|, |E_2|, \ldots, |E_m|) \in (\mathbb{R}^+)^m$.

Perimeter: $P(\mathcal{E}) = \mathcal{H}^{n-1}(\partial^* \mathcal{E})$, where $\partial^* \mathcal{E} = \bigcup \partial^* E_i$.

Notice: $\mathcal{P}(\mathcal{E}) = \frac{1}{2}\left(\sum_{i=1}^{m} P(E_i) + P\left(\cup_{i=1}^{m} E_i\right)\right)$

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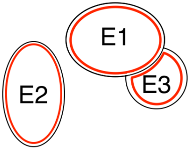

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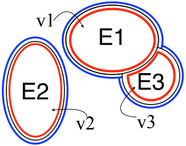

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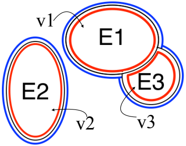

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Isoperimetric clustering problem

Given $\mathbf{v} = (v_1, \dots, v_m) \in (\mathbb{R}^+)^m$, consider

$$\inf \{\mathcal{P}(\mathcal{E}) : |\mathcal{E}| = \mathbf{v}\}.$$

Solutions are called *(m-)isoperimetric clusters*.

The classical isoperimetric clustering problem

(i) $\exists$ + regularity of solutions in $\mathbb{R}^n$, $n \geq 2$: [Almgren *Memoirs AMS* (1976)]
$\mathcal{E}$ minimal $\implies$ $\partial\mathcal{E}$ smooth outside singular set Σ, $\mathcal{H}^{n-1}(\Sigma) = 0$.

(ii) Structure of singularities Σ in $\mathbb{R}^2$: [Morgan (1994)]
(see [Taylor *Annals of Math* (1975)] for $n = 3$)
$\mathcal{E}$ minimal $\implies$ $\partial\mathcal{E} = \cup$ smooth cmc curves
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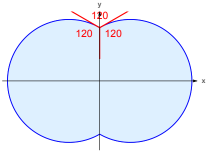

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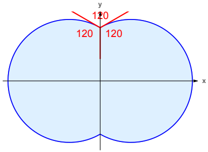
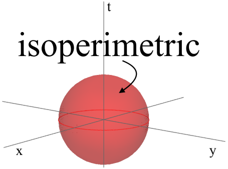


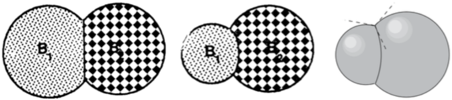
(iii) Classification of isoperimetric clusters?

(m = 1) Isoperimetric problem: solutions are balls.



Classification of isoperimetric clustering problem

(m = 2) Double bubble problem. Solutions are standard double bubbles:



Images from [Foisy & al. Pacific J. Math. (1993)] and [Maggi (book) (2012)]

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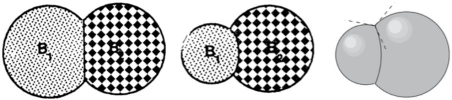

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(m = 3): If $n = 2$, solutions are the standard triple bubbles: [Wichiramala (2004)]

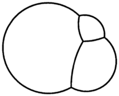

Image from [Wichiramala Crelle (2004)]

(m = 4) $n = 2$, same volumes [Paolini & Tortorelli *Calc. Var. PDEs* (2018)], [Paolini & Tamagnini *COCV* (2018)].

Examples of isoperimetric clusters in the plane

($\mathbf{m} \leq \min\{\mathbf{4}, \mathbf{n}\}$): [Milman&Neeman *preprint* (1 June 2022)]

Solution in $\mathbb{R}^n$ and $\mathbb{S}^n$ of the Multiple Bubble conjecture for
$m = 2$ - Double Bubble: $\forall n \geq 2$,
$m = 3$ - Triple Bubble: $\forall n \geq 3$
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("$\mathbf{m} = \infty$") Honeycomb theorem [Hales *Discrete & Comp. Geom.* (2001)]: a regular hexagonal grid is the best way to tassellate the plane into regions of equal area with the least total perimeter.

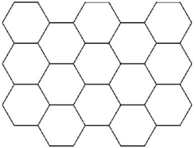


Image from Morgan's book

($\mathbf{m} \geq \mathbf{5}$) OPEN. Numerical simulations (*e.g.* [Cox, Morgan, Graner *Proc. Royal Soc. A* (2013)], [Del Nin *Adv. Calc. Var.* (2019)])

Today's subject

Planar clustering problem

$$\inf\left\{\mathcal{P}_h(\mathcal{E}) : |\mathcal{E}|_f = (v_1, \dots v_m)\right\},$$

$$\mathcal{P}_h(\mathcal{E}) = \frac{1}{2}\left(\sum_{i=1}^{m} P_h(E_i) + P_h(\cup_{i=1}^{m} E_i)\right),$$

for volume and perimeter with densities

$$P_h(E) = \int_{\partial^* E} h(x, \nu(x))\, d\mathcal{H}^1(x), \qquad |E|_f = \int_E f(x)\, dx.$$

$$h : \mathbb{R}^2 \times \mathbb{S}^1 \to (0, +\infty), \quad f : \mathbb{R}^2 \to (0, +\infty), l.s.c.$$

Examples:

- Gaussian plane $h(x) = f(x) = \frac{1}{2\pi} e^{-\frac{|x|^2}{2}}$.
  $\to$ [Milman & Neeman *Annals of Math.* (2022)] in dimension $n \geq 2$, with $m \leq n$.
- Wulff perimeters [Fonseca *Proc. Roy. Soc. London Ser. A* (1991)], [Morgan French Greenleaf *JGA* (1998)]
- A motivating example from sub-Riemannian geometry: the Grushin plane.

Grushin perimeter

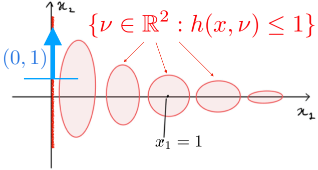
Denote $x = (x_1, x_2) \in \mathbb{R}^2$. For $a \geq 0$, $E \subset \mathbb{R}^2$ smooth set, let

$$P_{h_a}(E) := \int_{\partial E} \underbrace{\sqrt{\nu_1^2 + |x_1|^{2a}\nu_2^2}}_{h_a(x,\nu(x))} \, d\mathcal{H}^1(x).$$

☛ Related with the Heisenberg geometry.

☛ P_a is anisotropic, not translation invariant, not uniformly elliptic:

$h_a(x, \nu(x)) = 0$
if $x = (0, x_2)$ and $\nu = (0, \pm 1)$.



☛ There exists a transformation $\Phi : \mathbb{R}^2 \to \mathbb{R}^2$ such that

$$P_{h_a}(E) = P_{Eucl}(\Psi(E)) \; (h \equiv 1), \qquad \mathcal{L}^2(E) = |\Psi(E)|_{f_a}, \; f_a(x_1, x_2) = \frac{C}{|x_1|^{\frac{a}{a+1}}}.$$

→ Grushin double bubble

Today's subject

Planar clustering problem for volume and perimeter with densities

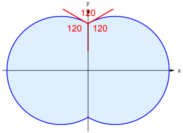
$$\inf\left\{\mathcal{P}_h(\mathcal{E}) : |\mathcal{E}|_f = (v_1, \dots v_m)\right\},$$

$$\mathcal{P}_h(\mathcal{E}) = \frac{1}{2}\left(\sum_{i=1}^{m} P_h(E_i) + P_h(\cup_{i=1}^{m} E_i)\right),$$

- **(m=1)** isoperimetric problem with density. [Cabré, Cañete, Cinti, De Philippis, Franzina, Fusco, Maggi, Miranda, Morgan, Pratelli, Rosales, Ros-Oton, Saracco, Serra,...]
- $\exists$ and regularity are in general not expected. Today, we do not focus on $\exists$.

? Are there assumptions under which minimizers are regular out of a small set?

What about the structure of the singular set depending on h, f?



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A preliminary version of our result is the following

Theorem (F., Pratelli, Stefani *Comm. Cont. Math* (2022))

Assume that

H1: *h is locally α-Hölder, for some $\alpha \in (0,1)$;*

H2: *f is loc. bdd., and the f-vol. of Eucl. balls satisfies the growth condition*

$$|B_{Eucl}(x,r)|_f \lesssim r^\eta, \quad \eta > 1/\beta, \quad r \ll 1, \qquad \beta = \frac{1}{2-\alpha}$$

Then the boundary of any isoperimetric cluster is a locally finite union of regular arcs joining in triple points at 120°.

- "*regular*" $= C^{1,\gamma}$ arcs with $\gamma = \frac{1}{2}\min\{\eta\beta - 1,\, \alpha\}$.
- (Growth): Euclidean, Gaussian: $\eta = 2$;
  Grushin ($h \equiv 1$, $f \equiv C_a|x_1|^{-\frac{a}{a+1}}$): $\eta = a + 2$.

Idea of the proof

Key point: show that multiple points are loc. finite & triple.
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1) Length bound in small balls for minimizers: $\mathcal{E}$ minimal $\implies \exists R \leq 1$:

$$\mathcal{H}^1(\partial^* \mathcal{E} \cap B(x,r)) < \frac{13}{2}\, r, \quad \forall r < R,\ x \in \mathbb{R}^2.$$

Proof. Let $\mathcal{E} =$ minimizer,
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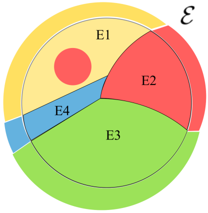

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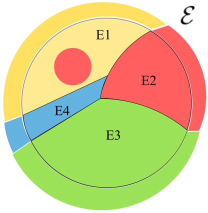

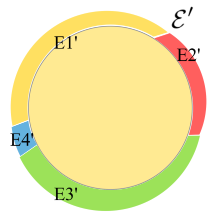

$$\Delta \mathcal{P} = \mathcal{P}(\mathcal{E}') - \mathcal{P}(\mathcal{E}) \leq 2\pi r \tilde{h}_{max} - \mathcal{P}(\mathcal{E}; B(x,r)), \tag{1a}$$

$$|\Delta V| \leq m |B(x,r)|_f. \tag{1b}$$

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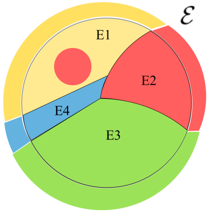

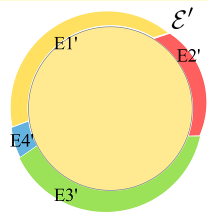

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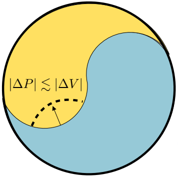
Need to repristinate the volume!

$\varepsilon - \varepsilon$ property



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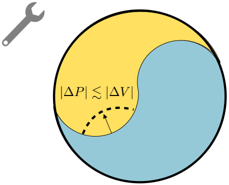

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$\varepsilon - \varepsilon^\beta$ property, $0 < \beta \leq 1$

$\mathcal{E}$ satisfyies the $\varepsilon - \varepsilon^\beta$ property if $\forall \varepsilon \in \mathbb{R}^m$, $|\varepsilon| \ll 1$, $\forall x \in \mathbb{R}^2$, there exist a cluster $\mathcal{F}$ and a radius R_β such that $\mathcal{F} \Delta \mathcal{E} \subset\subset \mathbb{R}^2 \setminus B(x, R_\beta)$ and

$$\underbrace{|\mathcal{F}|_f - |\mathcal{E}|_f}_{\Delta V} = \varepsilon\,, \quad \underbrace{\mathcal{P}_h(\mathcal{F}) - \mathcal{P}_h(\mathcal{E})}_{\Delta P} \lesssim |\varepsilon|^\beta.$$

☛ [Cinti,Pratelli *Crelle* (2012)], [Pratelli, Saracco *Adv Nonlin St* (2019)]:

f, h loc. bdd, $h = h(x, \nu)$ is α-Hölder in $x \implies$ any cluster of loc. finite perimeter satisfies the $\varepsilon - \varepsilon^\beta$ property with $\beta = \dfrac{1}{2-\alpha}$.

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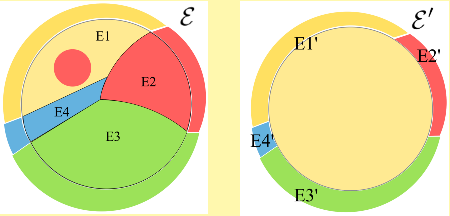

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$$|\Delta V| \le m|B(x,r)|_f = \varepsilon. \tag{1b}$$

Repristinate the volume via ($\varepsilon - \varepsilon^\beta$ prop.):

$$\implies \exists \mathcal{E}'' : |\mathcal{E}''|_f = |\mathcal{E}|_f, \ \mathcal{P}(\mathcal{E}'') \le \mathcal{P}(\mathcal{E}') + C|\varepsilon|^\beta \overbrace{\le \mathcal{P}(\mathcal{E}') + C r^{\eta\beta}}^{\text{growth } |B(x,r)|_f \le C_\eta r^\eta} . \tag{2}$$

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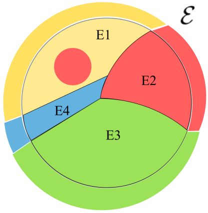

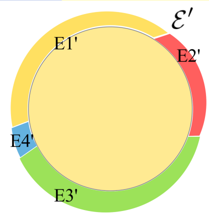

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By minimality: $\mathcal{P}(\mathcal{E}) \leq \mathcal{P}(\mathcal{E}'') \leq \mathcal{P}(\mathcal{E}) + 2\pi r \tilde{h}_{max} - \mathcal{P}(\mathcal{E}, B(x,r)) + Cr^{\eta\beta}$

□

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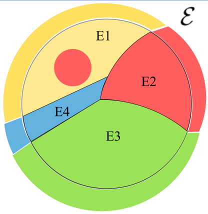

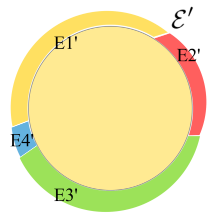

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$$\implies \exists \mathcal{E}'' : |\mathcal{E}''|_f = |\mathcal{E}|_f,\ \mathcal{P}(\mathcal{E}'') \leq \mathcal{P}(\mathcal{E}') + C|\varepsilon|^\beta \overbrace{\leq \mathcal{P}(\mathcal{E}') + Cr^{\eta\beta}}^{\text{growth } |B(x,r)|_f \leq C_\eta r^\eta} . \tag{2}$$

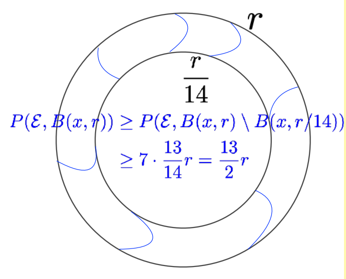
By minimality: $\cancel{\mathcal{P}(\mathcal{E})} \leq \mathcal{P}(\mathcal{E}'') \leq \cancel{\mathcal{P}(\mathcal{E})} + 2\pi r \tilde{h}_{max} - \mathcal{P}(\mathcal{E}, B(x,r)) + Cr^{\eta\beta}$

$$\mathcal{H}^1(\partial^*\mathcal{E} \cap B(x,r)) \leq \frac{\mathcal{P}(\mathcal{E}, B(x,r))}{\tilde{h}_{min}} \leq 2\pi r \frac{\tilde{h}_{max}}{\tilde{h}_{min}} + Cr^{\eta\beta} < \frac{13}{2}\, r.$$

□

2) The boundary of many small balls contains ≤ 3 points of $\partial^* \mathcal{E}$:

Proof. Start from ≤ 6.



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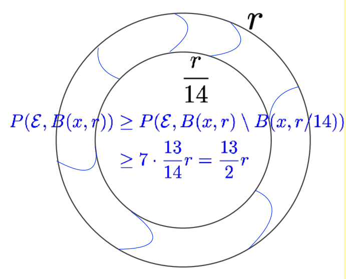
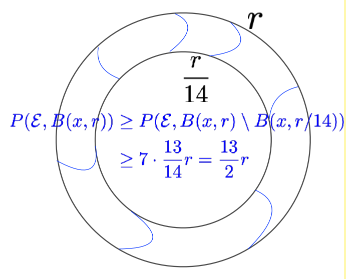
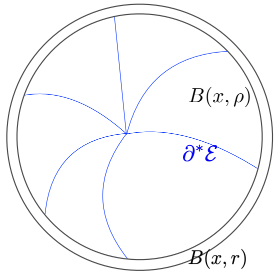


$$\exists R < 1 : \forall x \in \mathbb{R}^2, r \leq R,\ \exists \rho \in \left(\frac{r}{14}, r\right) : \sharp\left(\partial^*\mathcal{E} \cap \partial B(x,\rho)\right) \leq 6.$$

2) The boundary of many small balls contains ≤ 3 points of $\partial^* \mathcal{E}$:

Proof. Start from ≤ 6. $\rightarrow$ pass to ≤ 5 via Steiner 120°





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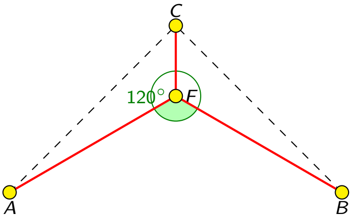
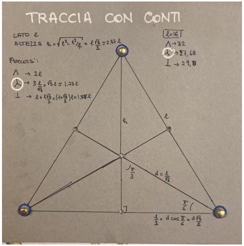
120° rule

If $A, B, C \in \mathbb{R}^2$ are three points such that $|A - C| = |B - C|$, and the angle $A\hat{C}B$ is $\leq 120°$, then there exists a unique point F minimizing the sum of the distances from the three points A, B, C and it has the 120° property.





From Math Circles @ Math UNIPD.

2) The boundary of many small balls contains ≤ 3 points of $\partial^*\mathcal{E}$

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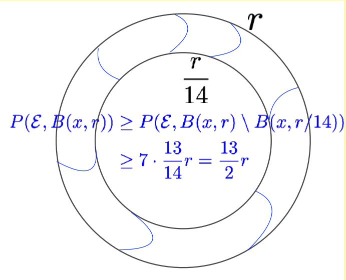

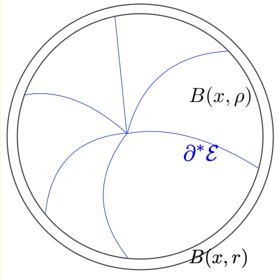

$$\exists R < 1 : \forall x \in \mathbb{R}^2, r \leq R, \ \exists \rho \in \left(\frac{r}{14}, r\right) : \sharp\,(\partial^*\mathcal{E} \cap \partial B(x,\rho)) \leq 6.$$

2) The boundary of many small balls contains ≤ 3 points of $\partial^* \mathcal{E}$

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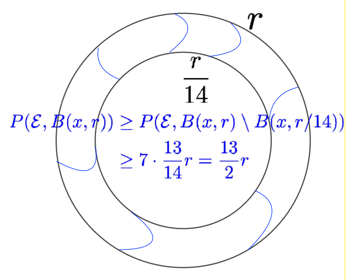
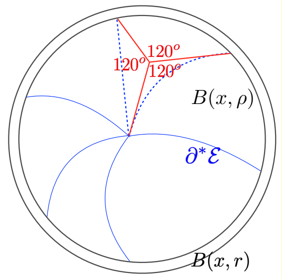
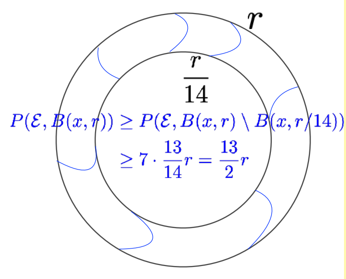


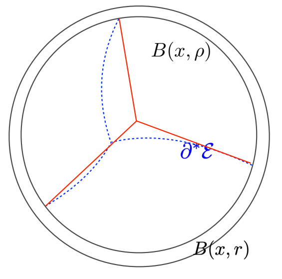


$$\exists R < 1, C > 0 : \forall x \in \mathbb{R}^2, r \leq R, \ \exists \rho \in \left(\frac{r}{C}, r\right) : \sharp \left(\partial^* \mathcal{E} \cap \partial B(x,\rho)\right) \leq 5.$$

2) The boundary of many small balls contains ≤ 3 points of $\partial^* \mathcal{E}$

Proof. Start from ≤ 6. $\rightarrow \cdots \leq 3$





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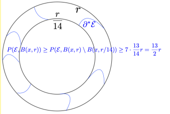
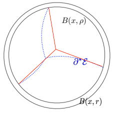
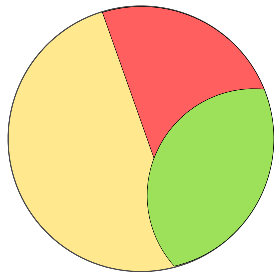


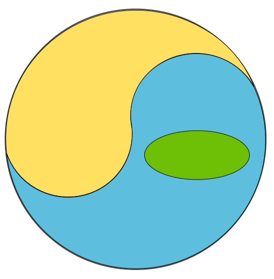
$\rightarrow$ pass to $\leq 5 \cdots \leq 3$



3) At most 3 colors at small scales



$\Leftarrow$ No-islands (cf. *no-infiltration in immiscible fluids* [Leonardi (2001)])



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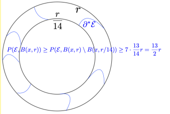

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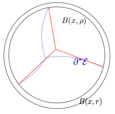

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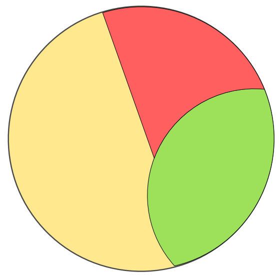

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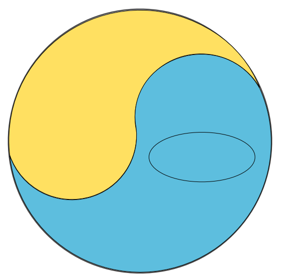
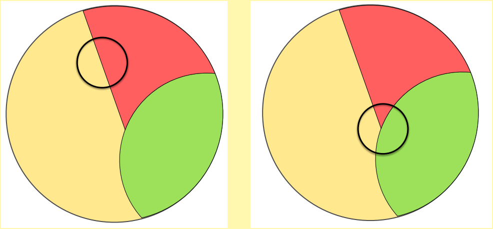


4) Distinguish 2 different cases



$\Longrightarrow$ Interface regularity in small balls and 120^o rule.

∗ 2 colors: geometric version of [Tamanini (1984)]

∗ 3-color points are a positive distance apart.

□

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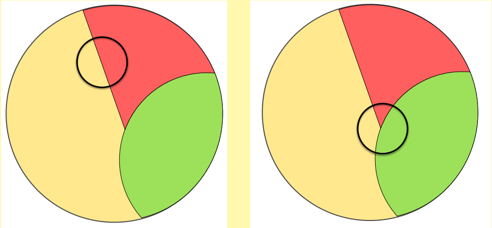

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□

We have used

(H0) The regularity of h for the C^1 regularity of the free boundary;
(H1) $\varepsilon - \varepsilon^\beta$ property;
(H2) Growth condition $|B(x,r)|_f \lesssim r^\eta$.

Our isotropic result

Theorem (F., Pratelli, Stefani *Comm. Cont. Math.* (2022))

Assume that

H0: *h is continuous, $\sqrt{h}$ is Dini continuous (i.e., $\int_0^1 \frac{\sqrt{\omega_h(t)}}{t}\,dt < +\infty$);*

H1: *$\mathcal{E}$ satisfies the $\varepsilon - \varepsilon^\beta$ property for $0 < \beta < 1$;*

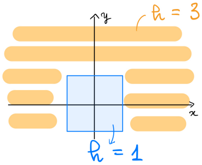
H2: *the f-volume of Euclidean balls satisfies the growth condition*

$$|B_{Eucl}(x,r)|_f \lesssim r^\eta, \quad \eta > 1/\beta, \quad r \ll 1.$$

Then the boundary of an isoperimetric cluster $\mathcal{E}$ is a locally finite union of C^1 arcs meeting in triple points at 120°.
If h is α-Hölder, the arcs are $C^{1,\gamma}$ with $\gamma = \frac{1}{2}\min\{\eta\beta - 1, \alpha\}$.

☛ The case $\eta\beta = 1$ can be also treated, implying C^1 regularity of the free boundary, under some extra assumptions ($t \mapsto C_\beta[t]$ is regular).

Otherwise it may fail. For example:
Euclidean volume $f \equiv 1$,



Outline

1. Intro: Isoperimetric clustering problem
2. First part: Isotropic perimeter density
3. Second part: The anisotropic case

Our anisotropic result

$$P_h(E) = \int_{\partial^* E} h(x, \nu(x)) \, d\mathcal{H}^1(x), \qquad |E|_f = \int_E f(x) \, dx.$$

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Theorem (F., Pratelli, Stefani *subm.* (2021))

Assume that

H0a: *h is continuous, $\sqrt{h}$ is Dini continuous in x;*

H0b: *h is C^1, strictly convex and uniformly round in ν;*

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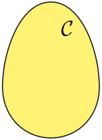
The assumptions: convexity and roundedness

H0b: *h is C^1, strictly convex and uniformly round in ν.*

Let $h(x, \lambda\nu) = \lambda h(x, \nu)$, $\lambda \geq 0$ and set $\mathcal{C}(x) := \{\nu \in \mathbb{R}^2 : h(x, \nu) \leq 1\}$.

Strict convexity: $\mathcal{C}$ is strictly convex for all $x \in \mathbb{R}^2$.

Uniform roundedness: the curvature of $\mathcal{C}(x)$ is bounded from below by $c > 0$ locally uniformly w.r.t. x.



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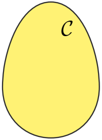


☛ Strict convexity is used both to prove that multiple junctions are locally finite triple points and for the regularity of the free boundary.

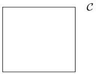
☛ Uniform roundedness is only used for the regularity of the free boundary.

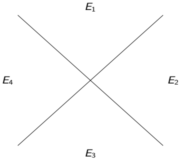
Regularity of the free boundary may fail already for $m = 1$ if h is not strictly convex and uniformly round.

The assumptions: quadruple points and C^1 regularity

H0b: h is C^1, strictly convex and uniformly round in ν.

☛ The fact that junction points are necessarily triple is related to the fact that $h \in \mathrm{C}^1$!





The unit ball $\mathcal{C}$ for $h(\nu) = \ell^1(\nu)$. Isoperimetric clusters may have quadruple junctions [Morgan,French,Greenleaf *JGA* (1998)].

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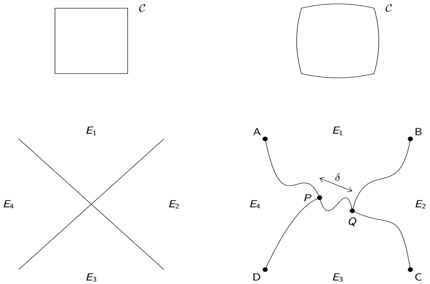


The unit ball $\mathcal{C}$ for $h(\nu) = \ell^1(\nu)$. Isoperimetric clusters may have quadruple junctions [Morgan,French,Greenleaf *JGA* (1998)].

On the right: the unit ball $\mathcal{C}$ for a modification of the ℓ^1 norm that gives h strictly convex and uniformly round, but not C^1.

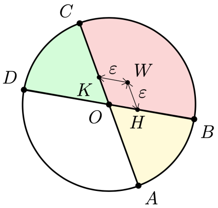
Quadruple points are still allowed in minimizers!

Comments

The anisotropic Steiner rule:

☛ Using that *h* is C^1 and strictly convex in ν, we prove that in small balls, no more than three radii can meet in order to minimize their anisotropic perimeter.

$\to$ [Alfaro et al *Pacific J Math* (1998)], [Lawlor & Morgan *Pacific J Math* (1994)] for minimizing networks.



$\longrightarrow$ Proof of the anisotropic result

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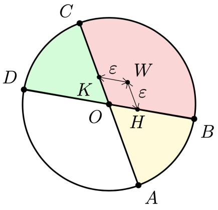


→ Proof of the anisotropic result

Directions at multiple points:

☛ Criticality: If $\tau_1, \tau_2, \tau_3 \in \mathbb{S}^1$ are the tangent directions to the boundary of an isoperimetric cluster meeting at a triple point O

$$\Longrightarrow \nabla \mathfrak{h}(\tau_1) + \nabla \mathfrak{h}(\tau_2) + \nabla \mathfrak{h}(\tau_3) = 0,$$

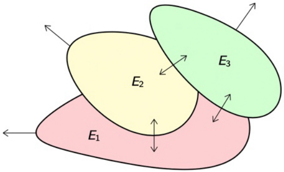
where $\mathfrak{h}$ is the perimeter density (computed on vectors rotated of 90 degrees) and "freezed" at the multiple point O.

☛ Number of admissible triples: depends on whether $h(x, \nu) = h(x, -\nu)$ or not.

The role of the symmetry of h

Perimeter: $P_h(E) = \int_{\partial^* E} h(x, \nu(x)) \, d\mathcal{H}^1(x)$

$$\mathcal{P}_h(\mathcal{E}) = \frac{1}{2}\left(\sum_{i=1}^{m} P_h(E_i) + P_h\left(\cup_{i=1}^{m} E_i\right)\right)$$



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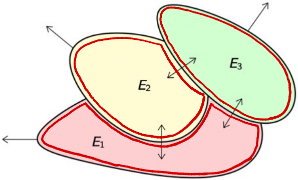

If $x \in \partial^* E_i \cap \partial^* E_j$: in $P(\mathcal{E})$ appears $\frac{1}{2}\left(\int_{\partial^* E_i} h(x,\nu)\, d\mathcal{H}^1 + \int_{\partial^* E_i} h(x,-\nu)\, d\mathcal{H}^1\right)$

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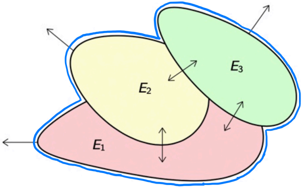


If $x \in \partial^* E_i \cap \partial^* E_j$: in $P(\mathcal{E})$ appears $\frac{1}{2}\left(\int_{\partial^* E_i} h(x,\nu)\, d\mathcal{H}^1 + \int_{\partial^* E_i} h(x,-\nu)\, d\mathcal{H}^1\right)$

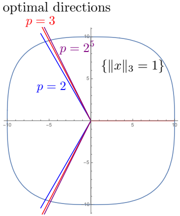
If $x \in \partial^* E_i \cap \partial^* E_0$: in $P(\mathcal{E})$ appears $\int_{\partial^* E_i} h(x,\nu)\, d\mathcal{H}^1$, where $E_0 = \mathbb{R}^2 \setminus \bigcup_{i=1}^{m} E_i$

The contributions coming from the "colored" sets may be different from the ones coming from the "white" one.

Possible directions at multiple points

> ? How many triples of directions are possible?

☛ If $h(x,\nu) = h(x,-\nu)$: Given $\tau_1 \in \mathbb{S}^1$, $\exists!(\tau_2,\tau_3)$



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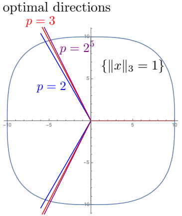

☛ If $h(x,\nu) = h(x,-\nu)$: Given $\tau_1 \in \mathbb{S}^1$, $\exists!(\tau_2, \tau_3)$

☛ If $h(x,\nu) \neq h(x,-\nu)$: Given $\tau_1 \in \mathbb{S}^1$, a triple containing τ_1 may exists and be unique, may not exists, or may be not unique.

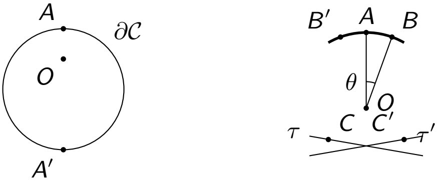

Figure: Left: an example with no admissible triple containing A. Right: an example with multiple admissible triples containing A.

Conclusions

$$P_h(E) = \int_{\partial^* E} h(x,\nu)\, d\mathcal{H}^1(x), \qquad |E|_f = \int_E f(x)\, dx.$$

- $\varepsilon - \varepsilon^\beta$ property and η-growth conditions are key for regularity.
- Dependence on the normal should be C^1 for having no more than 3 regular arcs meeting, together with some strict convexity assumptions.
- Symmtery/asymmetry plays a crucial role in discussing minimal triples of directions.
- We cover the Grushin setting → Grushin.
  We have a double bubble conjecture here.

Conclusions

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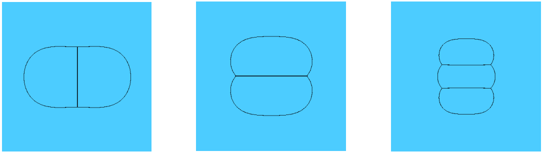

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INdAM Workshop

Rome 5–9/09/22

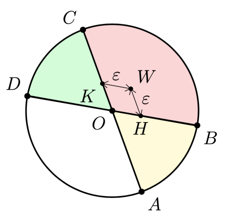
Registration is open!

Thank you for your attention!

Steps of the proof of the anisotropic result I

Key point: as in the isotropic case, show that multiple points are loc. finitely & triple.

1) The anisotropic Steiner rule. In small balls, no more than three radii can meet in order to minimize their anisotropic perimeter. (Replaces the 120° rule. Involves the C^1 regularity of *h* in the second variable and strictly convex.) [Alfaro et al *Pacific J Math* (1998)], [Lawlor & Morgan *Pacific J Math* (1994)]



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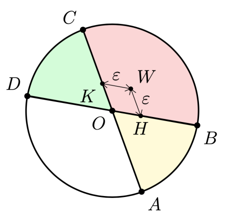

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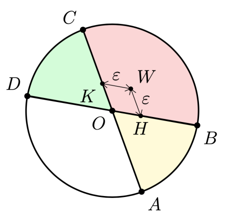

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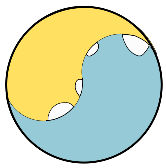
3) No-island (no-infiltration): only for colored chambers! ($i \neq 0$) We can still have some white holes between different chambers.



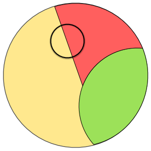
Steps of the proof of the anisotropic result II

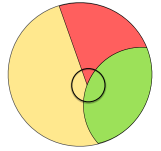
4) To exclude this case, we need to enter into the details of the reduced boundary $\partial^*\mathcal{E}$ and play with quasi-minimality, porosity (of colored regions), providing quantitative estimates of "errors" based on the strict convexity of h.

5) At small scales: many circles have ≤ 3 intersections with $\partial^*\mathcal{E}$

(it was Step 2 before!)

6) Conclusion





← Anisotropic result

The Grushin case

Grushin perimeter

Given $a \geq 0$, on a smooth set $E \subset \mathbb{R}^2$ it is the following:

$$P_a(E) = \int_{\partial E} \underbrace{\sqrt{\nu_1^2 + |x_1|^{2a}\nu_2^2}}_{h_a(x,\nu(x))} \, d\mathcal{H}^1(x), \quad x = (x_1, x_2).$$

☛ P_a is anisotropic, not translation invariant, not uniformly elliptic:

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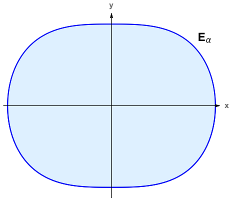
For $a = 1$, this is a section of the candidate isoperimetric set in the Heisenberg group ! [Pansu (1982)].



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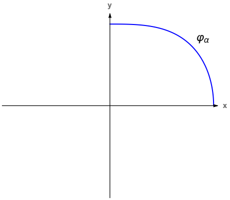

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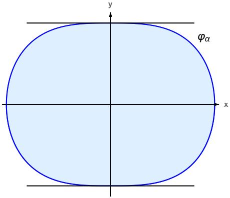

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Transformed plane

$$P_a(E) = P_{Eucl}(\Phi(E)) \qquad |E| = |\phi(E)|_{f_a} = \int_{\Phi(E)} \overbrace{|(a+1)x_1|^{-\frac{a}{a+1}}}^{f_\alpha(x)} \, dx.$$

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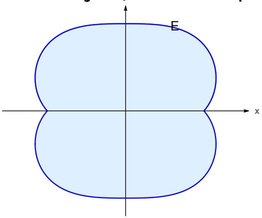

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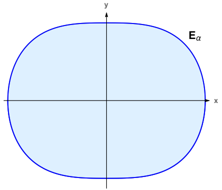

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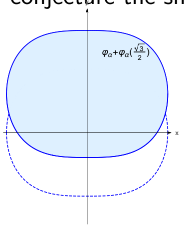

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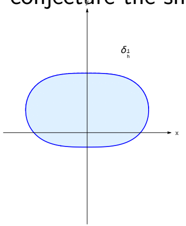

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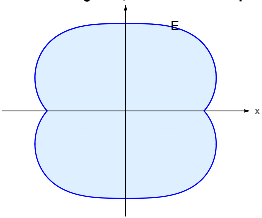
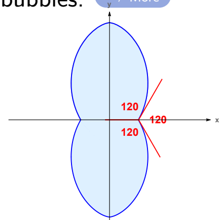




Multiple points are triple and satisfy the 120° rule in the transformed plane.

← Back

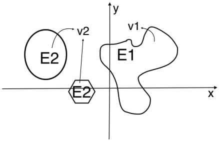
Grushin double bubble problem

$$(P_\alpha, |\cdot|): \; h = (\nu_1^2 + x^{2\alpha}\nu_2^2)^{1/2}, \; f \equiv 1.$$

(M=2): Grushin double bubble problem

$$\inf\left\{\mathcal{P}_\alpha(\mathcal{E}) : \mathcal{E} = \{E_1, E_2\} \in \mathcal{C}(v_1, v_2)\right\}$$

$$\mathcal{P}_\alpha(E) = \frac{1}{2}\Big(\mathcal{P}(E_1) + \mathcal{P}(E_2) + \mathcal{P}(E_1 \cup E_2)\Big).$$



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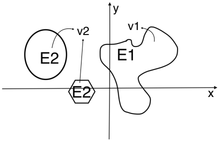


We consider the double bubble problem under more restrictive conditions:

(1) We assume $v_1 = v_2 = v \geq 0$.

(2) We assume specific structures of interfaces.

Problem 1: **(DBV)** Only vertical interfaces on $\{x_1 = 0\}$ allowed.

Problem 2: **(DBH)** Only horizontal interfaces allowed.

Main result

Theorem (F., Stefani (2019))

Let $v > 0$. Then solutions to problems **(DBV)**, **(DBH)** *exist.*

Moreover:

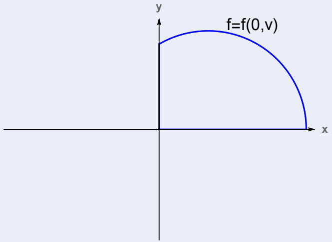
(DBV) $\mathcal{E} \subset \mathbb{R}^2$ *sol. to (DBV)* $\implies$ *up to vertical translations, we have*

$$\mathcal{E} = \{x \in \mathbb{R}^2 : |x_2| \leq f(|x_1|),\ |x_1| \leq r\},$$

$f \in C([0,r]) \cap C^\infty(]0,r[)$, $r \in]0,+\infty[$, *depends explicitly on v and α.*

In particular, if $\alpha > 0$, then $f'(0) = 0$.

Case $\alpha = 0$



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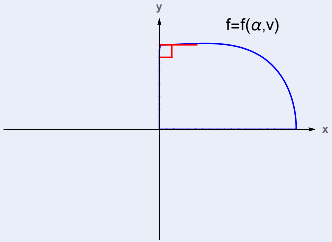

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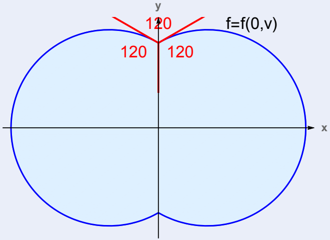

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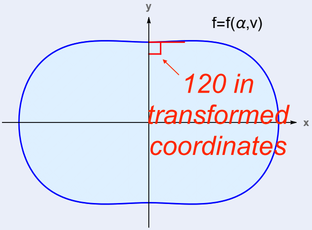

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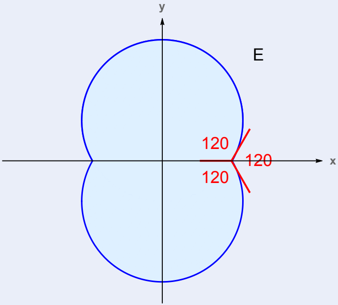

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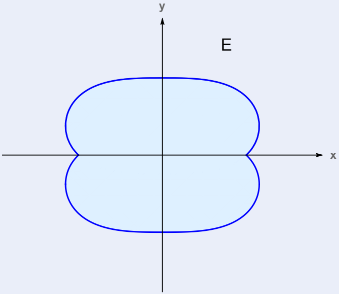

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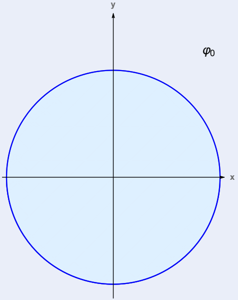

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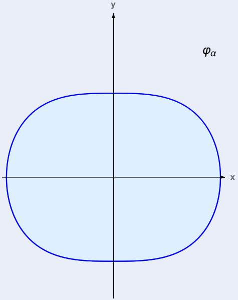

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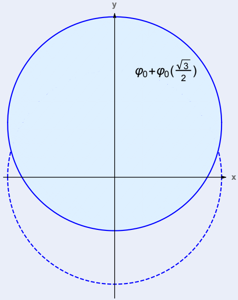

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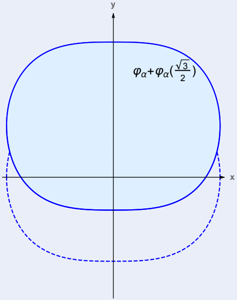

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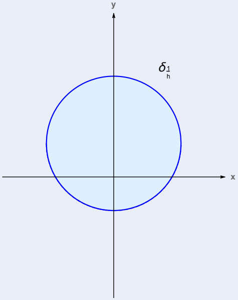

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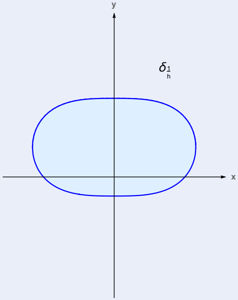

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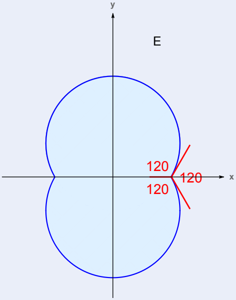
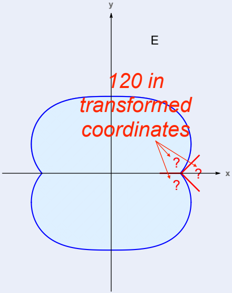
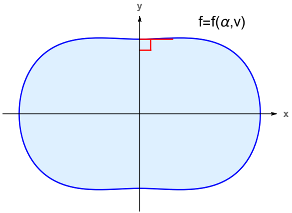


Case $\alpha = 1$

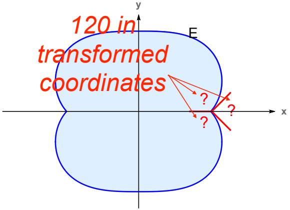


120 degrees rule.

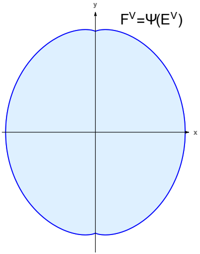
E^V solution to **(DBV)**.



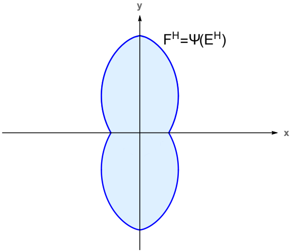
E^H solution to **(DBH)**.



The angle of interface between the bubbles is flat! Let $F^V = \Psi(E^V)$

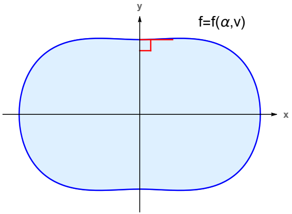


The angle of interface depends on α, v. Let $F^H = \Psi(E^H)$.

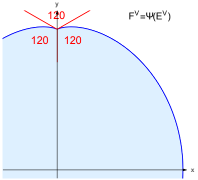


120 degrees rule.

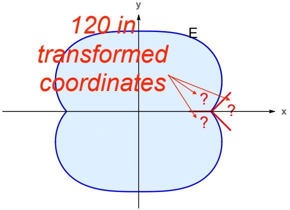
E^V solution to **(DBV)**.



The angle of interface between the bubbles is flat! Let $F^V = \Psi(E^V)$



E^H solution to **(DBH)**.



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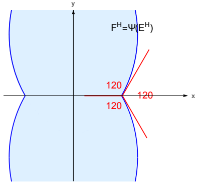


Corollary (F., Stefani)

The boundaries of the transformed bubbles meet at an angle of 120 *degrees.*

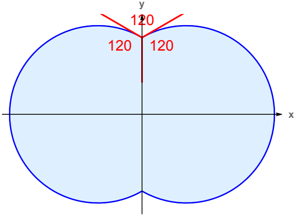
Comparison between vertical and horizontal

What can we conclude in view of the general double bubble problem?

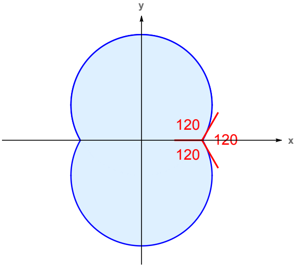
$$\inf\left\{\mathcal{P}_\alpha(E) : E = E_1 \cup E_2,\ \mathcal{L}^2(E_i) = v\right\}$$

($\alpha = 0$): Solutions to **(DBV)** and **(DBH)** are the standard double bubbles.

Vertical interface



Horizontal interface



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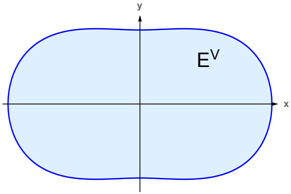
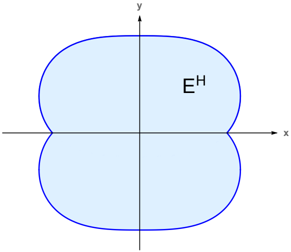


Horizontal interface



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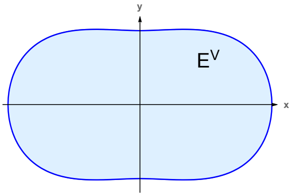
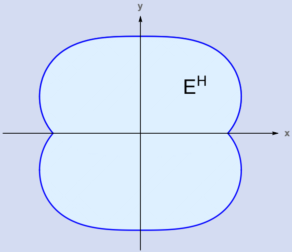


Horizontal interface



$$\mathcal{P}_1(E^V) \gneq \mathcal{P}_1(E^H)$$

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