Quantitative stability estimates for fractional inequalities

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(joint works with L. Brasco, R. Ognibene, B. Ruffini, and S. Vita)

An overview on stability estimates for geometric and functional inequalities

- The isoperimetric inequality;
- The Faber-Krahn inequality;
- The isocapacitary inequality.

An overview on stability estimates for geometric and functional inequalities

- **The isoperimetric inequality**

  The classical isoperimetric inequality states that balls minimize the perimeter functional (in the sense of De Giorgi) among all measurable sets with the same volume:

$$\frac{P(\Omega)}{|\Omega|^{\frac{n-1}{n}}} \geq \frac{P(B)}{|B|^{\frac{n-1}{n}}}.$$

  Question about stability: if a set Ω is almost optimal for the above inequality, can we say that it is "almost" a ball (in some sense)?

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Let us define the *Fraenkel asymmetry* of Ω:

$$\mathcal{A}(\Omega) = \min \left\{ \frac{|\Omega \Delta B|}{|\Omega|} \ : \ B \text{ is a ball with } |B| = |\Omega| \right\}$$

and the *isoperimetric deficit*

$$\delta_P(\Omega) = \frac{P(\Omega)}{|\Omega|^{\frac{n-1}{n}}} - \frac{P(B)}{|B|^{\frac{n-1}{n}}}.$$

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Theorem

There exists a constant $C(n)$ such that

$$\delta_P(\Omega) \geq C(n)\mathcal{A}(\Omega)^2.$$

Moreover, the exponent 2 is optimal.

Proofs by:

- Fusco, Maggi, Pratelli (2008): via symmetrization;
- Figalli, Maggi, Pratelli (2010): via mass transport;
- Cicalese,Leonardi (2012): via selection principle.

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- **The Faber-Krahn inequality**.

Let $\lambda_1(\Omega)$ denote the first eigenvalue of the Dirichlet-Laplacian, which has the following variational characterization:

$$\lambda_1(\Omega) := \inf\left\{ \int_\Omega |\nabla u|^2 \,:\, u \in C_0^\infty(\Omega),\, \int_\Omega u^2 = 1 \right\}.$$

Then

$$\lambda_1(\Omega)|\Omega|^{\frac{2}{n}} \geq \lambda_1(B)|B|^{\frac{2}{n}}$$

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$$\delta_{\lambda_1}(\Omega) = \lambda_1(\Omega)|\Omega|^{\frac{2}{n}} - \lambda_1(B)|B|^{\frac{2}{n}}.$$

Theorem (Brasco, De Philippis, Velichkov)

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- **The isocapacitary inequality**

Let cap(Ω) denote the capacity of the set Ω, that is

$$\mathrm{cap}(\Omega) := \inf\left\{ \int_{\mathbb{R}^n} |\nabla u|^2 \,:\, u \in C_0^\infty(\mathbb{R}^n),\, u \geq 1 \text{ on } \Omega \right\}.$$

Then

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$$\delta_{\text{cap}}(\Omega) = \text{cap}(\Omega)|\Omega|^{\frac{2}{n}-1} - \text{cap}(B)|B|^{\frac{2}{n}-1}.$$

Then, we have the following

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Proof of the Faber-Krahn inequality

We focus on the case of the Faber-Krahn inequality.

The proof of the inequality is based on radial decreasing rearrangements and the Pólya-Szegö inequality.

Let u^* be the radially symmetric decreasing function such that $|\{u^* > t\}| = |\{u > t\}|$, then we have:

$$\int |u^*|^2 = \int |u|^2$$

and

$$\text{Pólya-Szegö} \qquad \int |\nabla u^*|^2 \leq \int |\nabla u|^2.$$

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Let $\mu(t) = |\{u > t\}|$.We observe that

$$-\mu'(t) \geq \int_{\{u=t\}} \frac{d\mathcal{H}^{n-1}}{|\nabla u|}.$$

We have

$$\int_\Omega |\nabla u|^2 \overset{coarea}{=} \int_0^{+\infty} \left(\int_{\{u=t\}} |\nabla u|^2 \frac{1}{|\nabla u|} \, d\mathcal{H}^{n-1} \right) dt$$

$$\overset{Jensen}{\geq} \int_0^{+\infty} \left(\int_{\{u=t\}} |\nabla u| \frac{1}{|\nabla u|} \, d\mathcal{H}^{n-1} \right)^2 \frac{dt}{\int_0^{+\infty} \frac{d\mathcal{H}^{n-1}}{|\nabla u|}}$$

$$\geq \int_0^{+\infty} \frac{(P(\{u > t\}))^2}{-\mu'(t)} \, dt.$$

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The equality cases comes from the equality cases in the isoperimetric inequality.

An idea by Melas, Hansen and Nadirashvili: introduce quantitative elements in the proof of the Pólya-Szegö inequality, by using quantitative versions of the isoperimetric inequality.

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A quantitative (non optimal) Faber-Krahn inequality

Theorem (Brasco, De Philippis)

There exists an explicit dimensional constant $c_n > 0$ such that

$$\delta_{\lambda_1}(\Omega) \geq c_n \mathcal{A}(\Omega)^3.$$

Proof. We recall, from the proof of the Faber-Krahn inequality, that

$$\int_\Omega |\nabla u|^2 \geq \int_0^{+\infty} \frac{(P(\{u > t\}))^2}{-\mu'(t)}\, dt.$$

Moreover, by the quantitative isoperimetric inequality, we have

$$(P(\{u > t\}))^2 \geq (P(\{u^* > t\}))^2 + c\mu(t)^{\frac{2(n-1)}{n}} \mathcal{A}(\{u > t\})^2.$$

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Question: how can we pass from $\mathcal{A}(\{u > t\})$ to $\mathcal{A}(\Omega)$?

Idea: Choose a level T such that

$$\frac{|\{u > T\}\Delta\Omega|}{|\Omega|} \sim \mathcal{A}(\Omega), \quad \text{then} \quad \mathcal{A}(\{u > t\}) \gtrsim \mathcal{A}(\Omega) \quad \text{for every } 0 < t < T.$$

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Hence, we can deduce that

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Moreover, one can see that

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If $T \gtrsim \mathcal{A}(\Omega)$, we then deduce

$$\int_\Omega |\nabla u|^2 - \int_\Omega |\nabla u^*|^2 \geq \mathcal{A}(\Omega)^2 \frac{T^2}{\mathcal{A}(\Omega)|\Omega|}$$
$$\geq C\mathcal{A}(\Omega)^3.$$

It remains to deal with the case $T \ll \mathcal{A}(\Omega)$

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Case $T \ll \mathcal{A}(\Omega)$. In this case, we do not use the chain of inequality seen before, but we use a comparison argument, choosing as a competitor a suitable truncation of u (at level T). In this way, under the assumption that $T \ll \mathcal{A}(\Omega)$, one can conclude that

$$\lambda_1(\Omega) - \lambda_1(B) \geq C\mathcal{A}(\Omega).$$

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$$\lambda_1(\Omega) - \lambda_1(B) \geq C\mathcal{A}(\Omega).$$

The nonlocal setting

- The fractional isoperimetric inequality;
- The fractional Faber-Krahn inequality;
- The fractional isocapacitary inequality.

The fractional perimeter (Caffarelli, Roquejoffre, Savin)

Let $0 < s < 1/2$ and E be a bounded subset of $\mathbb{R}^n$. We define the s-perimeter of E as

$$\mathrm{Per}_s(E) = \int_E \int_{\mathbb{R}^n \setminus E} \frac{dx\, dy}{|x-y|^{n+2s}} = \frac{1}{2}[\chi_E]_{W^{2s,1}(\mathbb{R}^n)},$$

where χ_E denotes the characteristic function of the set E.

We have that

$$(1-2s)\mathrm{Per}_s(E) \to \mathrm{Per}(E), \quad \text{as } s \uparrow 1/2.$$

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The fractional isoperimetric inequality

Theorem (Almgren and Lieb, Frank and Seiringer)

$$\frac{Per_s(\Omega)}{|\Omega|^{\frac{n-2s}{n}}} \geq \frac{Per_s(B)}{|B|^{\frac{n-2s}{n}}}.$$

Equality holds if and only if Ω is a ball.

Question: What about stability?

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Let $\delta_{P_s}(\Omega)$ denote the fractional isoperimetric deficit

$$\delta_{P_s}(\Omega) = \frac{\mathrm{Per}_s(\Omega)}{|\Omega|^{\frac{n-2s}{n}}} - \frac{\mathrm{Per}_s(B)}{|B|^{\frac{n-2s}{n}}}.$$

Theorem (Fusco, Millot, Morini)

There exists a constant $C_{n,s} > 0$, such that

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The optimal quantitative fractional isoperimetric inequality

Theorem (Figalli, Fusco, Maggi, Millot, Morini)

There exists a constant $C_{n,s} > 0$, *such that*

$$\delta_{P_s}(\Omega) \geq C_{n,s} \mathcal{A}(\Omega)^2.$$

Remark

- the proof is based on a reduction to nearly spherical sets;
- it uses regularity theory for Λ-minimizers of the s-perimeter;
- the constant $C_{n,s}$ is not explicit.

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The fractional Faber-Krahn inequality

Let us consider now the eigenvalue problem for the fractional Laplacian.

For $0 < s < 1$, we consider the operator

$$(-\Delta)^s u(x) = \mathrm{PV} \int_{\mathbb{R}^n} \frac{u(x) - u(y)}{|x-y|^{n+2s}}\, dy.$$

The nonlocal quadratic functional associated to it is

$$[u]^2_{W^{s,2}(\mathbb{R}^n)} := \iint_{\mathbb{R}^n \times \mathbb{R}^n} \frac{|u(x) - u(y)|^2}{|x-y|^{n+2s}}\, dx\, dy.$$

We have that

$$[u]^2_{W^{s,2}(\mathbb{R}^N)} \sim \frac{C}{s} \int |u|^2\, dx, \qquad \text{for } s \searrow 0,$$

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We define the space $\mathcal{D}_0^{s,2}(\Omega)$ as the completion of $C_0^\infty(\Omega)$ with respect to the norm $[\cdot]_{W^{s,2}(\mathbb{R}^N)}$. The *first eigenvalue of the fractional Dirichlet-Laplacian of order s on* Ω (denoted by $\lambda_s(\Omega)$) is defined as the smallest real number λ such that the following boundary value problem

$$\begin{cases} (-\Delta)^s u &= \lambda\, u, & \text{in } \Omega, \\ u &= 0, & \text{in } \mathbb{R}^N \setminus \Omega, \end{cases}$$

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Similarly to the classical case, the first eigenvalue has the following variational characterization

$$\lambda_s(\Omega) = \min_{u \in \mathcal{D}_0^{s,2}(\Omega)} \left\{ [u]^2_{W^{s,2}(\mathbb{R}^N)} \; : \; \|u\|_{L^2(\Omega)} = 1 \right\}.$$

Theorem

We have

$$|\Omega|^{\frac{2s}{n}} \lambda_s(\Omega) \geq |B|^{\frac{2s}{n}} \lambda_s(B).$$

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Theorem (Brasco, C., Vita)

Let $0 < s < 1$ *and* $n \geq 2$. *For every open set* Ω *with finite measure, we have*

$$\delta_{\lambda_s} \geq \frac{C}{1-s} \mathcal{A}(\Omega)^{\frac{3}{s}},$$

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Remark

By letting $s \uparrow 1$, we recover the local quantitative Faber-Krahn inequality with exponent 3.

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The Caffarelli-Silvestre extension

Theorem

Given $u \in \mathcal{D}_0^{s,2}(\Omega)$, there exists a unique function E_u satisfying

$$\begin{cases} \operatorname{div}(z^{1-2s}\nabla E_u) = 0 & \text{in } \mathbb{R}^{n+1}_+ \\ E_u(x,0) = u(x) & \text{on } \mathbb{R}^n. \end{cases}$$

Moreover, U satisfies the following variational problem

$$\min \left\{ \int_{\mathbb{R}^{n+1}_+} z^{1-2s} |\nabla V|^2 \, dx\, dz \ : \ Trace(V) = u \right\},$$

and we have

$$[u]_{W^{s,2}(\mathbb{R}^n)} = \gamma_{n,s} \int_{\mathbb{R}^{n+1}_+} z^{1-2s} |\nabla E_u|^2 \, dx\, dz.$$

The quantitative fractional Faber-Krahn inequality

Proof of the main result. We try to follow the argument described for the local case and to apply it to the extension E_u. For almost every fixed $z > 0$, we define the function $E_u^*(\cdot, z)$ as the unique radially symmetric decreasing function on $\mathbb{R}^N$ such that for all $t > 0$

$$|\{x \in \mathbb{R}^N : E_u^*(x,z) > t\}| = |\{x \in \mathbb{R}^N : E_u(x,z) > t\}|.$$

Lemma (Fusco, Millot, Morini - Brasco, C., Vita)

We have

$$\int_{\mathbb{R}^{n+1}_+} z^{1-2s} |\nabla_x E_u|^2 \, dx \, dz \geq \int_{\mathbb{R}^{n+1}_+} z^{1-2s} |\nabla_x E_u^*|^2 \, dx \, dz,$$

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The quantitative fractional Faber-Krahn inequality

First, observe that, as a consequence of the previous lemma, we deduce the fractional Faber-Krahn inequality:

$$\begin{aligned}\lambda_s(\Omega) &= [u]^2_{W^{s,2}} = \gamma_{n,s} \int_{\mathbb{R}^{n+1}_+} z^{1-2s} |\nabla E_u|^2 \, dx \, dz \\ &\geq \gamma_{n,s} \int_{\mathbb{R}^{n+1}_+} z^{1-2s} |\nabla E^*_u|^2 \, dx \, dz \geq \gamma_{n,s} \int_{\mathbb{R}^{n+1}_+} z^{1-2s} |\nabla E_{u^*}|^2 \, dx \, dz \\ &= [u^*]^2_{W^{s,2}} = \lambda_s(B).\end{aligned}$$

The quantitative fractional Faber-Krahn inequality

Let us now try to insert quantitative elements, arguing as in the local case.

$$\iint_{\mathbb{R}^{n+1}_+} z^{1-2s} |\nabla_x E_u|^2 \, dx \, dz$$

$$\stackrel{coarea}{=} \int_0^{+\infty} z^{1-2s} \left(\int_0^{+\infty} \left(\int_{\{x \in \mathbb{R}^n \,:\, E_u(x,z)=t\}} |\nabla_x E_u|^2 \frac{d\mathcal{H}^{n-1}(x)}{|\nabla_x E_u|} \right) dt \right) dz$$

$$\stackrel{Jensen}{\geq} \int_0^{+\infty} z^{1-2s} \left(\int_0^{+\infty} \frac{P(\Omega_{t,z})^2}{\displaystyle\int_{\{x \in \mathbb{R}^n \,:\, E_u(x,z)=t\}} \frac{d\mathcal{H}^{n-1}(x)}{|\nabla_x E_u|}} \, dt \right) dz$$

where $\Omega_{t,z} = \{x \in \mathbb{R}^n \,:\, E_u(x,z) > t\}$ and $P(\Omega_{t,z})$ denotes the perimeter of the set $\Omega_{t,z}$.

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The quantitative fractional Faber-Krahn inequality

We now use the quantitative isoperimetric inequality (on the horizontal-superlevel sets of E_u), to get

$$\iint_{\mathbb{R}^{n+1}_+} z^{1-2\,s}\,|\nabla_x E_u|^2\,dx\,dz \geq \int_0^{+\infty} z^{1-2s}\left(\int_0^{+\infty} \frac{P(\Omega^*_{t,z})^2}{-\mu_z'(t)}\,dt\right)\,dz$$

$$+\;C_n\int_0^{+\infty} z^{1-2\,s}\left(\int_0^{+\infty} \frac{\left(\mu_z(t)^{\frac{N-1}{N}}\right)^2\,\mathcal{A}(\Omega_{t,z})^2}{-\mu_z'(t)}dt\right)\,dz$$

The quantitative fractional Faber-Krahn inequality

Problem: how to pass from $\mathcal{A}(\Omega_{t,z})$ to $\mathcal{A}(\Omega)$?

Roughly speaking, we will do this, in two steps:

- Relate the asymmetry of $\Omega_{t,z}$ to the one of $\Omega_t = \{x \in \Omega : u(x) > t\}$, i.e. something of the type

$$\mathcal{A}(\Omega_{t,z}) \simeq \mathcal{A}(\Omega_t), \qquad \text{for } t \ll 1 \text{ and } z \ll 1.$$

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Proposition

Let T *be such that* $|\{u > T\}\Delta\Omega| \sim \mathcal{A}(\Omega)$, *then*

$$\text{for } \frac{T}{4} \leq t \leq \frac{3}{8}\,T \qquad \text{and} \qquad \text{for } 0 < z \leq z_0 = c_{n,s}\left(\sqrt{\mathcal{A}(\Omega)\,|\Omega|}\;T\right)^{\frac{1}{s}},$$

we have

$$\Big|\Omega_{t,z}\Delta\Omega\Big| \leq \frac{1}{3}\,|\Omega|\,\mathcal{A}(\Omega), \tag{1}$$

and

$$\mathcal{A}(\Omega_{t,z}) \geq \frac{1}{5}\,\mathcal{A}(\Omega). \tag{2}$$

The quantitative fractional Faber-Krahn inequality

Remark

The proof of the proposition uses the trace estimate

$$\|E_u(\cdot, z) - u\|^2_{L^2(\mathbb{R}^n)} \lesssim z^{2s}.$$

Conclusion of the proof:

- if $T \gtrsim \mathcal{A}(\Omega)$: we conclude as in the local case, using the above proposition (observe also the dependence of z_0 on T and $\mathcal{A}(\Omega)$, that's why the final power depends on s!);
- if $T \ll \mathcal{A}(\Omega)$: again as in the local case, we use a comparison argument (comparing with a truncation of u). In this case, we do not need to use the extension and we work "downstairs".

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Some generalization

For $N \geq 2$ and $0 < s < 1$, we set

$$2_s^* = \frac{2\,N}{N - 2\,s}.$$

Then for every $1 \leq q < 2_s^*$, we consider the sharp Poincaré-Sobolev constant

$$\lambda_{s,q}(\Omega) = \min_{u \in \mathcal{D}_0^{s,2}(\Omega)} \left\{ [u]^2_{W^{s,2}(\mathbb{R}^N)} \,:\, \|u\|_{L^q(\Omega)} = 1 \right\}.$$

For $q \neq 2$, any solution of the variational problem above solves the following semilinear problem

$$\begin{cases} (-\Delta)^s u = \lambda_{s,q}(\Omega)\, |u|^{q-2}\, u, & \text{in } \Omega, \\ u = 0, & \text{in } \mathbb{R}^N \setminus \Omega. \end{cases}$$

Some generalization

For $N \geq 2$ and $0 < s < 1$, we set

$$2_s^* = \frac{2\,N}{N - 2\,s}.$$

Then for every $1 \leq q < 2_s^*$, we consider the sharp Poincaré-Sobolev constant

$$\lambda_{s,q}(\Omega) = \min_{u \in \mathcal{D}_0^{s,2}(\Omega)} \left\{ [u]^2_{W^{s,2}(\mathbb{R}^N)} \,:\, \|u\|_{L^q(\Omega)} = 1 \right\}.$$

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Some generalization

Theorem

Let $N \geq 2$, $0 < s < 1$ *and* $1 \leq q < 2_s^*$. *For every* $\Omega \subset \mathbb{R}^N$ *open set with finite measure, we have*

$$|\Omega|^{\frac{2}{q}-1+\frac{2s}{N}} \lambda_{s,q}(\Omega) - |B|^{\frac{2}{q}-1+\frac{2s}{N}} \lambda_{s,q}(B) \geq \frac{\sigma_1}{(1-s)} \mathcal{A}(\Omega)^{\frac{3}{s}},$$

for an explicit constant $\sigma_1 = \sigma_1(N,s,q) > 0$, *which is uniform as* $s \nearrow 1$.

Some generalization

Case $q = 1$: we call the quantity

$$\mathcal{T}_s(\Omega) := \frac{1}{\lambda_{s,1}(\Omega)} = \max_{u \in \mathcal{D}_0^{s,2}(\Omega)} \left\{ \left(\int_\Omega |u|\, dx \right)^2 \, : \, [u]^2_{W^{s,2}(\mathbb{R}^N)} = 1 \right\},$$

fractional torsional rigidity of order s of Ω.

Corollary

Let $N \geq 2$ and $0 < s < 1$. For every $\Omega \subset \mathbb{R}^N$ open set with finite measure, we have

$$\frac{\mathcal{T}_s(B)}{|B|^{\frac{N+2s}{N}}} - \frac{\mathcal{T}_s(\Omega)}{|\Omega|^{\frac{N+2s}{N}}} \geq \sigma_2\,(1-s)\,\mathcal{A}(\Omega)^{\frac{3}{s}},$$

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for an explicit constant $\sigma_2 = \sigma_2(N, s) > 0$, which is uniform as $s \nearrow 1$.

The case of regular sets

For regular sets $\Omega \subset \mathbb{R}^N$ we can slightly improve the exponent on the asymmetry, according to the following

Theorem

Let $N \geq 2$, $0 < s < 1$ and $1 \leq q < 2^*_s$. Let $\Omega \subset \mathbb{R}^N$ be an open bounded set, satisfying one of the following conditions:

A. *either $\partial\Omega$ is Lipschitz and Ω satisfies the exterior ball condition, with radius ρ;*

B. *or $\partial\Omega$ is $C^{1,\alpha}$, for some $0 < \alpha < 1$.*

Then we have

$$|\Omega|^{\frac{2}{q}-1+\frac{2s}{N}}\lambda_{s,q}(\Omega) - |B|^{\frac{2}{q}-1+\frac{2s}{N}}\lambda_{s,q}(B) \geq \frac{C}{1-s}\mathcal{A}(\Omega)^{2+\frac{1}{s}}.$$

The case of regular sets

Idea: thanks to the regularity results by Ros-Oton and Serra, we know that u is of class $C^s(\mathbb{R}^N)$, and this allows us to upgrade the L^2 control

$$\|E_u(\cdot, z) - u\|_{L^2(\mathbb{R}^N)} \lesssim z^s, \qquad \text{for } z > 0 \tag{3}$$

to an L^∞ control

$$\|E_u(\cdot, z) - u\|_{L^\infty(\mathbb{R}^N)} \lesssim z^s, \qquad \text{for } z > 0. \tag{4}$$

The fractional isocapacitary inequality

We consider the fractional generalization of the capacity, defined for compact sets as follows

$$\mathrm{Cap}_s(\Omega) = \inf\left\{[u]_s^2\colon\, u \in C_c^\infty(\mathbb{R}^n),\ u \geq 1 \text{ on } \Omega\right\}, \tag{5}$$

Again, as a sconsequence of the fractional Pólya-Szëgo type inequality, one can derive the fractional version of the isocapacitary inequality, stating that

$$|\Omega|^{(2s-n)/n}\mathrm{Cap}_s(\Omega) \geq |B|^{(2s-n)/n}\mathrm{Cap}_s(B).$$

The quantitative fractional isocapacitary inequality

We define the fractional isocapacitary deficit as

$$\delta_{\mathrm{Cap}_s}(\Omega) := |\Omega|^{(2s-n)/n}\mathrm{Cap}_s(\Omega) - |B|^{(2s-n)/n}\mathrm{Cap}_s(B).$$

then we have,

Theorem (C.,Ognibene, Ruffini)

$$\delta_{\mathrm{Cap}_s}(\Omega) \geq C_{n,s}\mathcal{A}(\Omega)^{\frac{3}{s}}.$$

Thanks a lot for your attention!