Introduction to isoperimetry in Riemannian manifolds and the emergence of nonsmooth structures

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Outline of the talk

- Introduction to manifolds with lower bounds on the Ricci curvature, nonnegative Ricci curvature and Euclidean Volume Growth

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- The sharp isoperimetric inequality in such class, an elementary heuristic proof, its issues
- The asymptotic mass decomposition for minimizing sequences for the isoperimetric problem and the emergence of nonsmooth spaces

Nonnegative Ricci curvature

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and that

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If Ricci bounded from below by k

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Nonnegative Ricci curvature if $k = 0$.

Nonnegative Ricci: Bishop Gromov and AVR

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Asymptotic Volume Ratio.

We say (M, g) has Euclidean Volume Growth if $\mathrm{AVR}(g) > 0$.

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RMK: the above inequality is the one that holds true also on Riemannian cones, where $\mathrm{AVR(g)}$ can be characterized by the aperture (Morgan-Ritoré).

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- 2022 - Antonelli- Pasqualetto-Pozzetta-Semola (preprint) Inequality and full rigidity in RCD spaces, in particular rigidity for finite perimeter sets in smooth manifolds.

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We recall that the isoperimetric profile is defined as

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so that

$$I(|E|) = P(E).$$

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We have

$$\Delta r(x) = \mathrm{H}_{\partial E_{r(x)}}(x),$$

where $E_R = E \cup \{r \leq R\}$.

Ingredients in the proof

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$$\Delta r_{|\partial E} = \mathrm{H}_{\partial E}.$$

A sharp bound on the Laplacian of r

Let $r(x) = \mathrm{dist}(x, E)$.
Applying the Bochner identity

$$0 = \frac{1}{2}|\nabla r|^2 \geq |\nabla\nabla r|^2 + \langle \nabla \Delta r, \nabla r\rangle = \sum_{i,j=1}^{n-1} (\nabla\nabla r(e_i, e_j))^2 + \partial_r \Delta r$$

$$\geq \sum_{j=1}^{n-1} \frac{(\nabla\nabla r(e_j, e_j))^2}{n-1} + \partial_r \Delta r$$

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Integrating, and coupling with

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we get

$$\Delta r \leq \frac{\mathrm{H}_{\partial E}}{1 + \frac{\mathrm{H}_{\partial E}}{n-1} r}.$$

By the Divergence Theorem and the coarea formula

$$P(E_R) - P(E) = \int_{E_R \setminus E} \Delta r = \int_0^R \int_{\partial E_s} \Delta r$$

Estimates on the growth of sublevel sets of r

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Dividing both sides by $\omega_n R^n$, recalling that

$$0 < \mathrm{AVR}(g) = \lim_{R \to +\infty} \frac{|B(o,R)|}{\omega_n R^n}$$

we obtain

$$\mathrm{H}_{\partial E} \geq (n-1) \left(\frac{|\mathbb{S}^{n-1}| \mathrm{AVR}(g)}{P(E)} \right)^{\frac{1}{n-1}}.$$

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"True" for any isoperimetric set E of volume V.

Let I the isoperimetric profile, E_V isoperimetric of volume V. We have

$$I'(V) = \mathrm{H}_{E_V} \geq (n-1)\left(\frac{|\mathbb{S}^{n-1}|\mathrm{AVR}(g)}{P(E_V)}\right)^{\frac{1}{n-1}} = (n-1)\left(\frac{|\mathbb{S}^{n-1}|\mathrm{AVR}(g)}{I(V)}\right)^{\frac{1}{n-1}}.$$

Conclusion of the proof

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Integrating we get

$$\frac{P(E_V)^n}{|E_V|^{n-1}} \geq \frac{P_{\mathbb{R}^n}(\mathbb{B})^n}{|\mathbb{B}|^{n-1}}\mathrm{AVR}(g),$$

and thus for any bounded set Ω with volume V we have

$$\frac{P(\Omega)^n}{|\Omega|^{n-1}} \geq \frac{P_{\mathbb{R}^n}(\mathbb{B})^n}{|\mathbb{B}|^{n-1}}\mathrm{AVR}(g).$$

Main issues

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- How to deal with regularity issues?

Let E_i have all volume $V > 0$: $|E_i| = V$ and

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is E_i converging to some isoperimetric set of volume V?

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Ritoré-Rosales (04, TAMS): On any complete Riemannian manifold, and for any volume V, we can always find a minimizing sequence $\{\Omega_i\}_{i\in\mathbb{N}}$ with $|\Omega_i| = V$ such that

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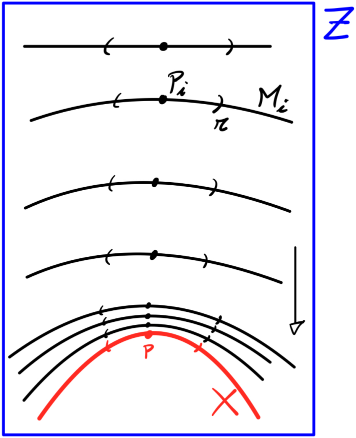
What happens to Ω_i^d???

Gromov-Hausdorff convergence

Definition (pmGH-convergence)

Take (M_i, g_i) and $p_i \in M_i$. The sequence of pointed mms $(M_i, \mathrm{dist}_i, \mathscr{H}_i^n, p_i)$ *pmGH-converges* to a pointed mms $(X, \mathrm{dist}, \mathscr{H}^n, p)$ if there is a complete separable (Z, d) and isometric embeddings $\iota_i : M_i \hookrightarrow Z$, $\iota : X \hookrightarrow Z$ s.t.

- $\iota_i(B_r(p_i)) \to \iota(B_r(p))$ in Hausdorff distance in Z for any $r > 0$,
- $\mathscr{H}_i^n(B_r(p_i)) \to \mathscr{H}^n(B_r(p))$ for any $r > 0$.



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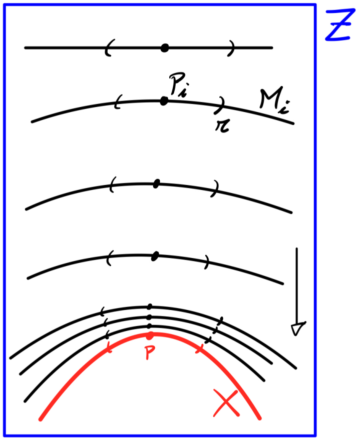

We are going to look at the convergence of sequences like $(M, \mathrm{dist}_g, \mathscr{H}_d^n, x_i)$ with x_i drifting away at infinity.

Spaces at infinity

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X is a special pointed metric measure space $(X, \mathrm{dist}, \mathscr{H}^n{}_{\mathrm{dist}_g}, x)$ with a notion of $Ric \geq kg$.

We are going to look at the convergence of sequences like $(M, \mathrm{dist}_g, x_i)$ with x_i drifting away at infinity, with $Ric \geq kg$ for some nonpositive k and such that $B(x_i, 1) > v > 0$ (noncollapsed)

FACT: Nonnegative Ricci and $AVR(g) > 0 \Rightarrow$ noncollapsed.

No compactness in the pmGH-topology in the smooth class...
...but compactness in a bigger class

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$$\frac{1}{2}\Delta|\nabla f|^2 \geq \frac{(\Delta f)^2}{n} + \langle \nabla \Delta f, \nabla f \rangle + k|\nabla f|^2$$

Theorem (Asymptotic mass decomposition (contributions below))

Let (M, g) with $Ric \geq kg$ and $|B(p,1)| > c > 0$. Let $V > 0$. Then some volume $0 \leq V_1 \leq V$ is recovered by a bounded isoperimetric set Ω_1 in M.

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The runaway volume is recovered as follows. There exists a finite number of j's such that there exists a sequence of $\{\Omega^d_{i,j}\}_{i\in\mathbb{N}}$ and RCD spaces $(X_j, \mathrm{dist}_j, \mathscr{H}^n_{\mathrm{dist}_j}, x_j)$ appearing as pmGH limits of $(M, \mathrm{dist}_g, x_i)$ such that $\Omega^d_{i,j}$ converges in volume and perimeter to an isoperimetric set $Z_j \subset X_j$.

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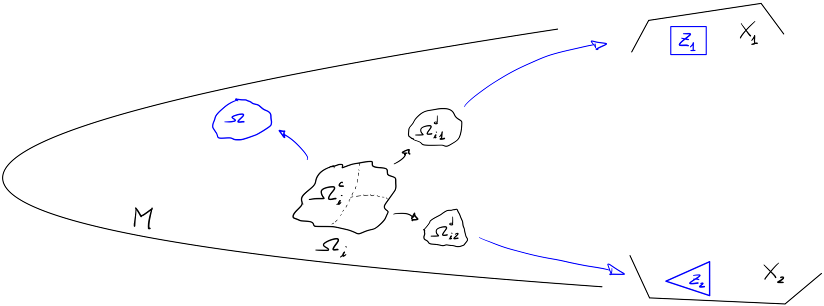
In particular

$$I_M(V) = \inf\{P(E)\, \text{such that}\, E \subset M\, \text{with}\, |E| = V\} = P_M(\Omega_1) + \sum_j P_{X_j}(Z_j).$$

Asymptotic mass decomposition



Contributions to the asymptotic mass decomposition

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- Antonelli, Nardulli, Pozzetta (2022, preprint), ultimate result in the generality of RCD $(X, d, \mathscr{H}_d)$ spaces.

Topic of Lecture 2!

What's coming next

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But what about (as sharp as possible) conditions ensuring the existence of isoperimetric sets?

Applying everything we deduce that

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On manifolds with nonnegative Ricci, $\mathrm{AVR}(g) > 0$ and certain conditions on the asymptotic cone at infinity, there exist isoperimetric sets of any volume big enough. (Lecture 6).

Thank you