Isoperimetric problems

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An isoperimetric problem with strong capacitary repulsion

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Based on joint works with Michael Goldman, Cyrill Muratov and Matteo Novaga

Literature

- Goldman - Novaga - R. Existence and stability for a non-local isoperimetric problem of charged liquid drops
  ARMA 2015
- Muratov - Novaga - R. On equilibrium shapes of charged flat drops
  CPAM 2018
- " Conductive flat drops in a confining potential
  ARMA 2022
- Goldman - Novaga - R. Reifenberg flatness for almost minimizers of the perimeter under minimal assumptions
  PAMS 2022
- " Rigidity of the ball for an isoperimetric problem with strong capacitary repulsion
  PREPRINT 2022

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Plan of the talk

- The isoperimetric problem
- Physical & mathematical motivations
- A general approach
- Old results
- New results
- Proofs

1/ The isoperimetric problem

$$\min \left\{ P(E) + \frac{Q^2}{\mathrm{cap}_\alpha(E)} \;\middle|\; \begin{array}{l} |E| = m \\ E \subset \mathbb{R}^n \end{array} \right\}$$

where **P** is the Caccioppoli-De Giorgi perimeter

and

$$\mathrm{cap}_\alpha(E)^{-1} = \inf \left\{ \iint \frac{d\mu(x)\, d\mu(y)}{|x-y|^{n-\alpha}} \;\middle|\; \mu(E) = 1 \right\}$$

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We are (mostly) interested in the cases

$$\alpha \in [0,1]$$

2/ Physical & mathematical motivations

Mathematically the model energy takes the form

$E \subset \mathbb{R}^n$, $|E| = m$

$\mathcal{A}(E) + \mathcal{R}(E)$

INSTANCES OF MODELS

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INSTANCES OF MODELS

$P(E) \longrightarrow V_\alpha(E) = \int_E \int_E \frac{1}{|x-y|^{n-\alpha}} \, dx \, dy$

OHTA-KAWASAKI LIMITING ENERGY

• GAMOW DROP MODEL

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$\longrightarrow V_G(E) = \int_E \int_E G(x,y) \, dx \, dy$

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$E \subset \mathbb{R}^n$, $|E| = m$

$$A(E) + R(E)$$

INSTANCES OF MODELS

$P(E) \longrightarrow$

- $V_\alpha(E) = \iint_{E\,E} \frac{1}{|x-y|^{n-\alpha}}\, dx\, dy$ — OHTA-KAWASAKI LIMITING ENERGY • GAMOW DROP MODEL
- $V_G(E) = \iint_{E\,E} G(x,y)\, dx\, dy$
- $Q^2 / \mathrm{cap}_\alpha(E)$ — LORD RAYLEIGH CHARGED DROPS MODEL
- Wasserstein-type repulsions — LIPID BILAYERS

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INSTANCES OF MODELS

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$$\int_{\partial E} H^2$$

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INSTANCES OF MODELS

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$$\int_{\partial E} H^2$$

$\int |\nabla u|^2$ → $\iint \frac{u(x)^p\, u(y)^p}{|x-y|^{n-\alpha}}\, dx\, dy$ — HARTREE TYPE ENERGIES

λ_1 or other spectral fctnals

3/ Relevant questions

By scaling (or merely by a proper setting of the parameters) the problems translate into

$$A + \varepsilon R$$

MOSTLY CORRESPONDING TO A POSITIVE POWER OF THE MASS m

As $\varepsilon = 0$ implies $\exists$ of minimizers, usually the ball and $\varepsilon \sim +\infty$ suggests $\nexists$ of minimizers, one is led to ask whether

4/ For $\varepsilon \sim 0$ is the ball a (stable?) minimizer?

5/ Is there $M > 0$ s.t. for $\varepsilon > M$, non-existence of minimizers occurs?

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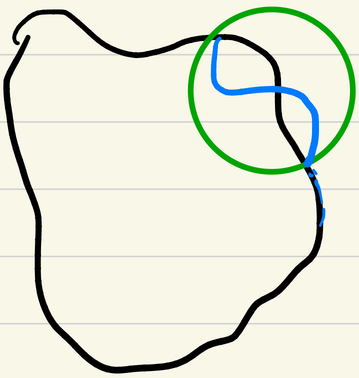
WE FOCUS ON QUESTION 4/

4/ A general strategy for the existence issue

[KNÜPFER–MURATOV CPAM 2013/2014
FIGALLI–FUSCO–MAGGI–MILLOT–MORINI CMP 2015]

. For $\mathcal{E} = P(E) + \varepsilon \iint_{E\,E} \frac{dx\,dy}{|x-y|^{n-\alpha}}$

i/ Notice that V_α acts as a volume term



E: —
F: —

$E \Delta F \subset B_r(x)$

If E is a minimizer,

$$P(E) - P(F) \leq \varepsilon \left(V_\alpha(F) - V_\alpha(E) \right)$$
$$\lesssim \varepsilon\, r^n$$

i.e. a minimizer of $\mathcal{E}$ is a quasi-minimizer of P.

ii/ If $F = B$,

$$|E \Delta B|^2 \overset{(*)}{\lesssim} P(E) - P(B) \lesssim \varepsilon \quad \text{i.e.}$$

a minimizer is ε-close to a ball in L^1

FUSCO–MAGGI–PRATELLI ANN. OF MATH. 2008
(*) CICALESE–LEONARDI ARMA 2012
FIGALLI–MAGGI–PRATELLI INVENT. MATH. 2010

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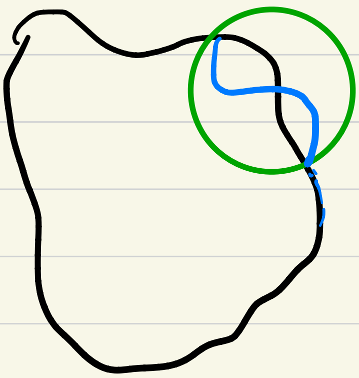

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a minimizer is ε-close to a ball in L^1 → $C^{1,\alpha}$

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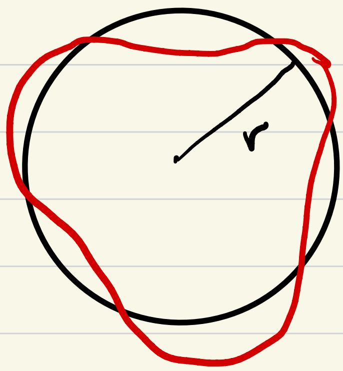
iii/ ... hence

$$\partial E = \left\{ (r + \varphi(x)) \frac{x}{|x|} \;\middle|\; x \in \partial B_r \,,\ \varphi : \partial B_r \to \mathbb{R} \right\}$$



i.e., E is a nearly spherical set

iv) Prove stability in the class of nearly spherical sets ..

$$|E \Delta B|^2 \lesssim P(E) - P(B) \lesssim \varepsilon \left(V_\alpha(B) - V_\alpha(E) \right) \lesssim \varepsilon |E \Delta B|^2$$

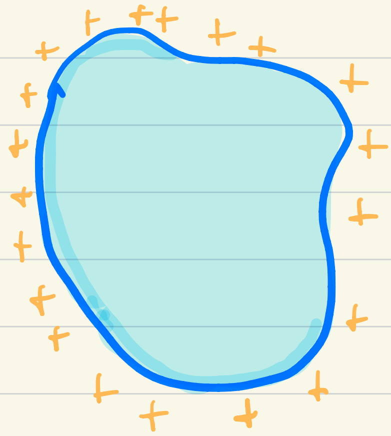
by choosing ε small enough.

5/ The charged liquid drop model

We recall now the energy we are interested in

$$\mathcal{E} = P(E) + Q^2 / \mathrm{cap}_\alpha(E)$$

__Remark__: for $n=3$, $\alpha=2$, this describes the energy of a charged liquid droplet (LORD REYLEIGH, 1882)



A minimizer μ of

$$\mathrm{cap}_\alpha^{-1}(E) = \inf \iint \frac{d\mu(x)\, d\mu(y)}{|x-y|^{n-\alpha}} \quad , \mu(E)=1$$

is the EQULIBRIUM MEASURE (HARMONIC) of E.

__FACTS__:
- If $n-\alpha \leq n-2$, $\mathrm{spt}\,\mu \subset \partial E$
- If $n-\alpha > n-2$, $\mathrm{spt}\,\mu = \bar{E}$
- If $E=B$, and $n-\alpha \leq n-2$ then

$$\mu = \text{const.}\, \mathcal{H}^{n-1} \llcorner \partial B$$

- If $n-\alpha = n-2$ & $\mu = \text{const.}\, \mathcal{H}^{n-1} \llcorner \partial E$, then $E=B$

If $n-\alpha \neq n-2$, open problem

6/ First results

Theorem (Goldman - Novaga - R. 2015)

- Let $n-\alpha < n-1$, $n \geq 2$. Then

$$\min \left\{ P(E) + Q^2 / \mathrm{cap}_\alpha(E) \;\middle|\; |E| = m \right\}$$

does not admits solution.

Precisely, the l.s.c. relaxation in L^1 of

$P + Q^2 \mathrm{cap}_\alpha^{-1}$ is P.

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- An interesting regularization proposal (Muratov-Novaga, Proc. R.S.A. 2016)

Replace $\mathrm{cap}_\alpha(E)^{-1}$ with

$$\mathrm{cap}_\alpha^\varepsilon(E)^{-1} = \min_{\mu(E)=1} \left\{ \iint_{E\,E} \frac{d\mu(x)\, d\mu(y)}{|x-y|^{n-\alpha}} + \varepsilon \int \mu^2(x)\, dx \right\}$$

Theorem $\cdot$ $\exists$ of minimizers holds for

$$\min \left\{ P(E) + Q^2 / \mathrm{cap}_\alpha^\varepsilon(E) \;\middle|\; |E| = m \right\}$$ exist, (Muratov-Novaga 2016)

and regularity too (De Philippis-Hirsch-Vescovo, CMP, 2019)

7/ $n-\alpha \geq n-1$: the case $n=2$, $\alpha=1$

A first positive result (FLAT DROPS)

Theorem: Let $n=2$, $\alpha=1$. Then there exists an (explicit) $Q_0>0$ s.t. if $Q\leq Q_0$, the ball is the only minimizer of $P+Q^2/\mathrm{cap}_1$.

If $Q>Q_0$ there are no minimizers.

(MURATOV-NOVAGA-R. CPAM 2018)

Furthermore, the l.s.c. envelope of $P+Q^2/\mathrm{cap}_1$ is given by

$$\mathcal{F}(E)=\begin{cases} P(E)+Q^2/\mathrm{cap}_1(E) & \text{if } Q\leq \pi^2\mathrm{cap}_1(E) \\ P(E)+\pi\left(Q-\mathrm{cap}_1(E)\right) & \end{cases}$$

(MURATOV-NOVAGA-R ARMA 2022)

8/ $n-\alpha \geq n-1$: the general case

Theorem (GOLDMAN-NOVAGA-R. PREPRINT 2022)

Let $n \geq 2$, $n \geq n-\alpha \geq n-1$. There exists $Q_0 > 0$ s.t. if $Q < Q_0$ then the ball is the only minimizer of $P + Q^2/\mathrm{cap}_\alpha$.

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MAIN IDEAS IN THE PROOF

- Consider first $n-\alpha < n-1$ and let

$$\mathcal{E}_\varepsilon(E) = P(E) + Q^2 / cap_\alpha^\varepsilon(E)$$

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MAIN IDEAS IN THE PROOF

- Consider first $n-\alpha < n-1$ and let

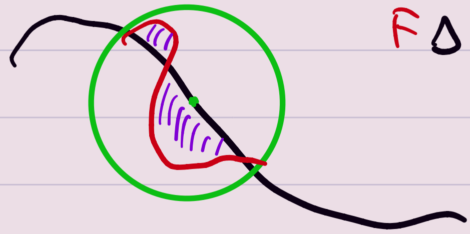
$$\mathcal{E}_\varepsilon(E) = P(E) + \frac{Q^2}{\mathrm{cap}_\alpha^\varepsilon(E)}$$

- Show existence of GENERALIZED MINIMIZERS, i.e. a "set" $(E_i)_{i \in \mathbb{N}} \subset (\mathbb{R}^n)^{\mathbb{N}}$ s.t. $\sum |E_i| = m$ and $\sum_i \mathcal{E}_\varepsilon(E_i) \leq \sum_i \mathcal{E}_\varepsilon(F_i)$ $\forall (F_i)_i$ s.t. $\sum_i |F_i| = m$

- Show that any E_i satisfies

$$P(E_i) \leq P(F) + C \cdot r^{n-\alpha}$$

Relevant inequality:



$$(\mathrm{cap}_\alpha^\varepsilon)^{-1}(F) - (\mathrm{cap}_\alpha^\varepsilon)^{-1}(E) \lesssim (\mathrm{cap}_\alpha^\varepsilon)(E \Delta F)$$

$$\lesssim r^{n-\alpha}$$

· Show that any E_i satisfies

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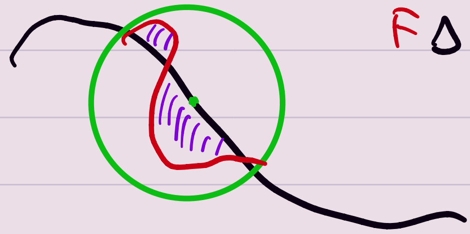

$$(\mathrm{cap}_\alpha^\varepsilon)^{-1}(F) - (\mathrm{cap}_\alpha^\varepsilon)^{-1}(E) \lesssim (\mathrm{cap}_\alpha^\varepsilon)(E \Delta F)$$

$$\lesssim r^{n-\alpha}$$

· For $\alpha \in (0,1)$ this implies UNIFORM density estimates entailing:

- $\exists$ of CLASSICAL minimizers
- $C^{1,\gamma}$-regularity of minimizers.

Hence: send $\varepsilon \to 0$ and, via the quantitative isoperimetric inequality and a perturbative argument, conclude that the ball is the only minimizer.

9/ The case $\alpha = 1$

If $\alpha = 1$, the quasi-minimization

$$P(E) \leq P(F) + \Omega^2 r^{n-1}$$

does not implies $C^{1,\alpha}$ regularity. But...

9/ The case $\alpha = 1$

If $\alpha = 1$, the quasi-minimization

$$P(E) \leq P(F) + Q^2 r^{n-1} \quad , \quad E \Delta F \subset B_r \quad (*)$$

does not implies $C^{1,\alpha}$ regularity. But...

Theorem (Ambrosio - Paolini 1999, Goldman - Novaga - R 2022)

There exists $Q_0 \ll 1$ s.t. if $Q < Q_0$ and E is a quasiminimizer in the sense of $(*)$, then E is a Reifenberg-flat set.

Moreover, one can improve the quasi-minimality inequality as

$$P(E) \leq P(F) + C\left(\left(\int_{B_r} \mu_E^{\frac{2n}{n+1}}\right)^{\frac{n+1}{n}} + r^n\right), \quad E \Delta F \subset B_r$$

Hence IF $\left(\int_{B_r} \mu_E^{\frac{2n}{n+1}}\right)^{\frac{n+1}{n}} \lesssim r^{n-\gamma}$ for some $\gamma < 1$

the required regularity holds!

Theorem (GOLDMAN – NOVAGA – R. 2022)

There exists $Q_0 > 0$ s.t. if $Q < Q_0$ then

$$\left(\int_{B_r(x)} \mu_E^{\frac{2n}{n+1}} \right)^{\frac{n+1}{n}} \lesssim r^{n-\gamma} \quad , \text{ where } x \in \partial E ,$$

$$\gamma = \gamma(\alpha, n, Q) < 1$$

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$$\gamma = \gamma(\alpha, n, Q) < 1$$

MAIN STEPS IN THE PROOF

1/ A regularity result

Following TERRACINI – VERZINI – ZILIO JEMS 2016 we show that the capacitary potential u_E, solving

$$\begin{cases} (-\Delta)^{1/2} u_E = 0 & \text{on } \mathbb{R}^n \setminus E \\ u_E = \text{const.} & \text{on } E \\ u_E(\infty) = 0 \end{cases} \quad \text{or} \quad \begin{cases} (-\Delta)^{1/2} u_E = \mu_E \\ u_E(\infty) = 0 \end{cases}$$

is Hölder continuous up to ∂E, so that

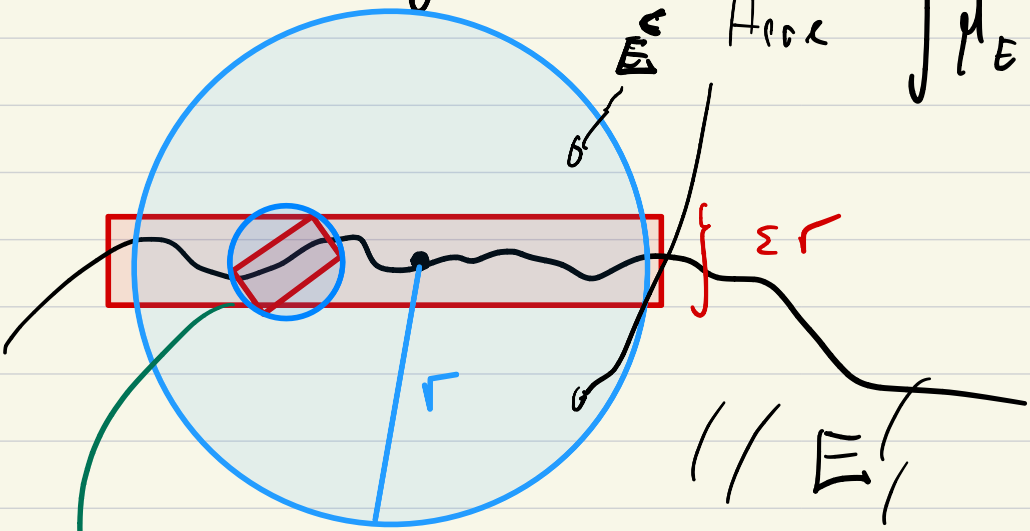
$$\mu_E = (-\Delta)^{1/2} u_E \lesssim \text{dist}(\cdot, \partial E)^{-(1-\eta)}$$

for some $\gamma > 0$.

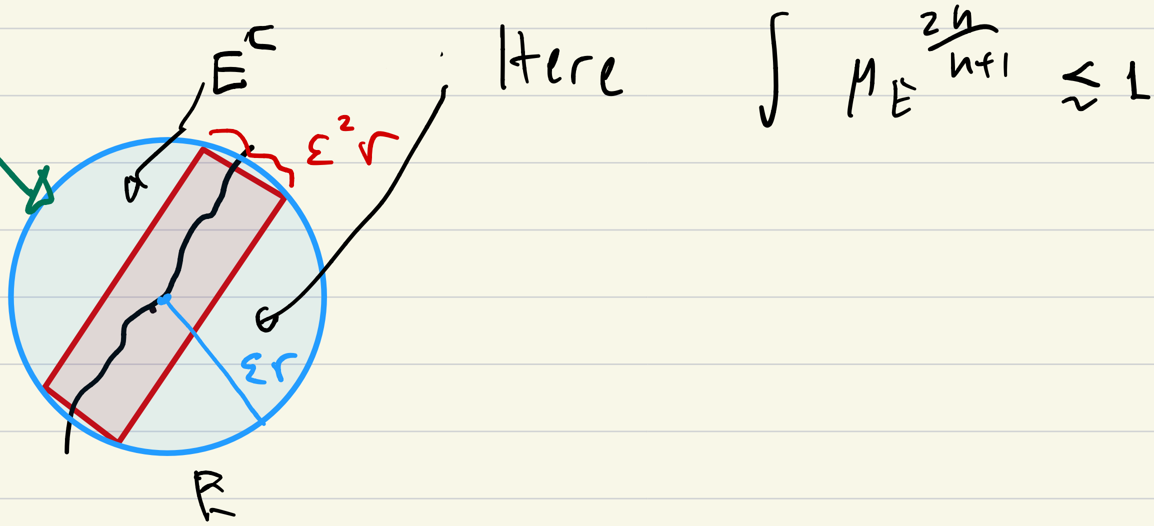
2/ A covering argument

As $\mu_E \lesssim \operatorname{dist}(\cdot, \partial E)^{-(1-\gamma)}$ and E is Reifenberg flat, one can cover E with shrinking balls

1.



2.



⋮

Iterate and conclude via a recovering argument.

Thank you
for the
attention

Thank you for the attention

...and sorry for being lenghty