Isoperimetric Problems

Alessandra Pluda

Steiner Problem, global and local minimizers of the length functional

Università di Pisa

Structure of the talk

Steiner Problem : classical formulation
basic properties

Minimal partitions problem : relation with Steiner
paired calibrations
a local minimality result

Currents with coefficients in a group : relation with Steiner
calibrations vs. paired calibrations
a global minimality result

Łojasiewicz - Simon inequality for minimal networks

Steiner Problem

Let $\mathcal{C} = \{p_1, \dots, p_n\}$ be a finite collection of points in $\mathbb{R}^2$.

Steiner Problem: Find a connected set K such that $\mathcal{C} \subset K$ and the length of K is minimal

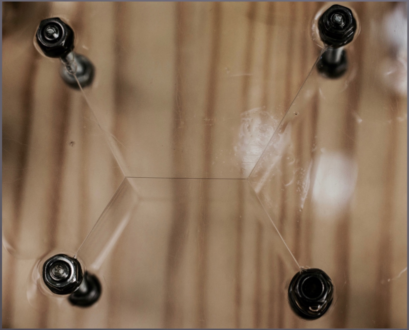
$$\boxed{\inf\{\mathcal{H}^1(K) : K \subseteq \mathbb{R}^2, \text{ connected and such } \mathcal{C} \subseteq K\}}$$

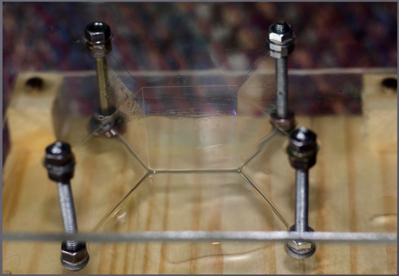
The exists a minimizer to the Steiner problem

Steiner Problem

Let $\mathcal{C} = \{p_1, \dots, p_n\}$ be a finite collection of points in $\mathbb{R}^2$.

Steiner Problem: Find a connected set K such that $\mathcal{C} \subset K$ and the length of K is minimal

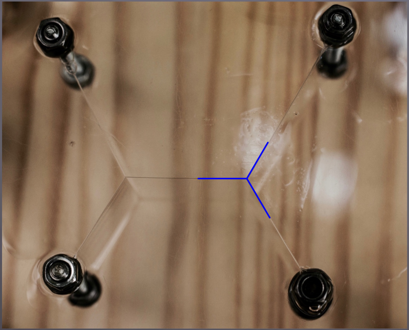




Steiner Problem

Let $\mathcal{C} = \{p_1, \dots, p_n\}$ be a finite collection of points in $\mathbb{R}^2$.

Steiner Problem: Find a connected set K such that $\mathcal{C} \subset K$ and the length of K is minimal



Minimizers are networks without loops composed of straigth segments meeting at triple junctions forming angles of $120°$

Steiner Problem

Let $\mathcal{C} = \{p_1, \ldots, p_n\}$ be a finite collection of points in $\mathbb{R}^2$.

Steiner Problem : Find a connected set K such that $\mathcal{C} \subset K$ and the length of K is minimal

Minimizers are not necessary unique

Networks

A network can be seen as a couple $\mathcal{N}=(G,\Gamma)$

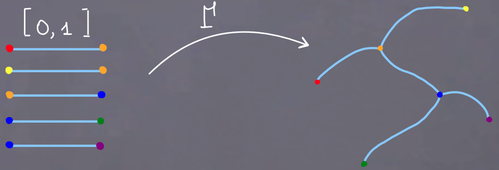
where G is an "abstract graph" with some identifications

and $\Gamma : G \to \mathbb{R}^2$ is a map $\Gamma = (\gamma^1, \dots, \gamma^m)$

fixed topology

$\Updownarrow$

fixed G



A network is degenerate if $\exists\, i \in \{1, \dots, m\}$ such that $L(\gamma^i) = 0$

A network composed of straigth segments that meet at triple junctions forming angles of 120° is called minimal

Minimal networks are local minimizers

Minimal networks are local minimizers of the lenght functional

Minimal networks are local minimizers

Minimal networks are local minimizers of the lenght functional

in the sence that

if $\mathcal{N}_* = (G, \Gamma_* = (\gamma^1_*, \dots, \gamma^N_*))$ is a minimal network

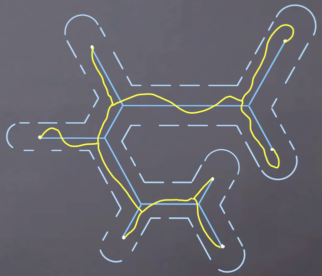
then there exists $\delta > 0$ such that

$$L(\mathcal{N}) \geq L(\mathcal{N}_*)$$

whenever $\mathcal{N} = (G, \Gamma = (\gamma^1, \dots, \gamma^N))$

is a triple junction network

such that $\| \gamma^i \cdot \sigma^i - \gamma^i_* \|_{C^0} < \varepsilon$



Fixed topology

The minimizers have at most $m = n - 2$ triple junctions.

Once we fix the topology (we fix the underlying graph G)
the problem reduces to determine the location
of the triple junctions $x_1, \dots, x_m \in \mathbb{R}^2$

$$\inf \left\{ \mathcal{L}(x_1, \dots, x_m) = \sum |x_i - p_i| + \sum |x_i - x_j| \;\middle|\; x_1, \dots, x_m \in \mathbb{R}^2 \right\}$$

The problem is convex. Existence of minimizer is trivial.

Minimizers can be degenerate

Fixed topology

Minimizers can be degenerate

Example :

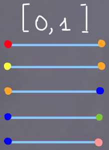
DATA

p_1 •

p_2 •

p_3 •

p_4 •



UNKNOWN

x_1 •

x_2 •

$$\inf \{ \mathcal{L}(x_1, x_2) = |x_1 - p_1| + |x_1 - p_2| + |x_2 - p_3| + |x_2 - p_4| + |x_1 - x_2| \text{ with } x_1, x_2 \in \mathbb{R}^2 \}$$

Fixed topology

Minimizers can be degenerate

Example :

DATA

p_1

p_2

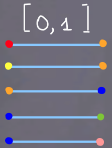
p_3

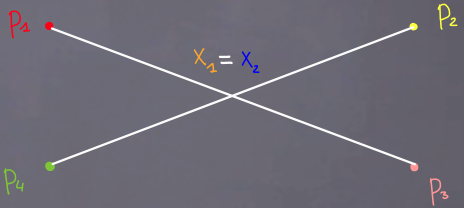
p_4

UNKNOWN

x_1

x_2





$$\inf \{ \mathcal{L}(x_1, x_2) = |x_1 - p_1| + |x_1 - p_2| + |x_2 - p_3| + |x_2 - p_4| + |x_1 - x_2| \text{ with } x_1, x_2 \in \mathbb{R}^2 \}$$

Fixed topology

The minimizers have at most $m = n - 2$ triple junctions.

Once we fix the topology (we fix the underlying graph G) the problem reduces to determine the location of the triple junctions

$$\boxed{\inf \left\{ \mathcal{L}(x_1, \dots, x_m) = \sum |x_i - p_i| + \sum |x_i - x_j| \;\middle|\; x_1, \dots, x_m \in \mathbb{R}^2 \right\}}$$

The problem is convex. Existence of minimizer is trivial.

The number of possible topologies is finite

Minimize among all minimizers with fixed topology

Difficulty the number of possible topologies escaletes as the number of points $p_1, \dots, p_n$ increases

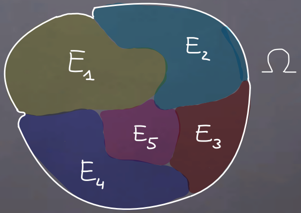
Minimal partitions

Let $n \in \mathbb{N}$ and let $\Omega \subseteq \mathbb{R}^2$ be an open set.
We say that $\mathbf{E} = (E_1, \dots, E_n)$ is a Caccioppoli partition of Ω if

$$E_i \subseteq \Omega, \quad |E_i \cap E_j| = 0, \quad |\Omega \setminus \bigcup_{i=1}^{n} E_i| = 0 \quad \text{and} \quad P(E_i, \Omega) < +\infty .$$

Minimal partitions

Let $n \in \mathbb{N}$ and let $\Omega \subseteq \mathbb{R}^2$ be an open set.
We say that $\mathbf{E} = (E_1, \dots, E_n)$ is a Caccioppoli partition of Ω if
$E_i \subseteq \Omega$, $|E_i \cap E_j| = 0$, $|\Omega \setminus \bigcup_{i=1}^{n} E_i| = 0$ and $P(E_i, \Omega) < +\infty$.



Minimal partitions

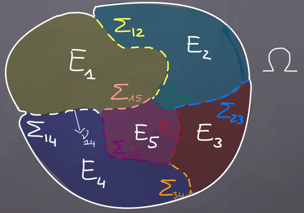
Let $n \in \mathbb{N}$ and let $\Omega \subseteq \mathbb{R}^2$ be an open set.
We say that $\mathbf{E} = (E_1, \dots, E_n)$ is a Caccioppoli partition of Ω if
$E_i \subseteq \Omega$, $|E_i \cap E_j| = 0$, $|\Omega \setminus \bigcup_{i=1}^{n} E_i| = 0$ and $P(E_i, \Omega) < +\infty$.

We denote by $\Sigma_{ij} = (\partial^* E_i \cap \partial^* E_j)$

ν_i outer outward normal to E_i

$\nu_{ij} = \nu_i = -\nu_j$ unit normal to Σ_{ij} pointing from E_i to E_j

$\mathcal{P}(\mathbf{E}) = \frac{1}{2}\sum_{i=1}^{n} P(E_i, \Omega)$



Minimal partitions

Let $n \in \mathbb{N}$ and let $\Omega \subseteq \mathbb{R}^2$ be an open set.
We say that $\mathbf{E} = (E_1, \dots, E_n)$ is a Caccioppoli partition of Ω if
$E_i \subseteq \Omega$, $|E_i \cap E_j| = 0$, $|\Omega \setminus \bigcup_{i=1}^{n} E_i| = 0$ and $P(E_i, \Omega) < +\infty$.

We denote by $\Sigma_{ij} = (\partial^* E_i \cap \partial^* E_j)$
ν_i outer outward normal to E_i
$\nu_{ij} = \nu_i = -\nu_j$ unit normal to Σ_{ij} pointing from E_i to E_j
$\mathcal{P}(\mathbf{E}) = \frac{1}{2} \sum_{i=1}^{n} P(E_i, \Omega)$

Minimal partitions problem : given $\tilde{\mathbf{E}} = (\tilde{E}_1, \dots, \tilde{E}_n)$ find

$$\inf \{ \mathcal{P}(\mathbf{E}) : \mathbf{E} = (E_1, \dots, E_n) \text{ Caccioppoli partition of } \Omega \text{ such that } \mathrm{tr}_\Omega \chi_{E_i} = \mathrm{tr}_\Omega \chi_{\tilde{E}_i} \}$$

Equivalence of Steiner and Minimal Partitions Problem

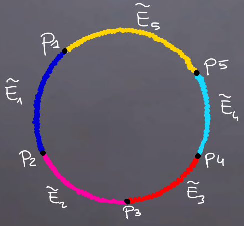
Let Ω be convex.
There exists a minimizer to the Minimal partitions problem.

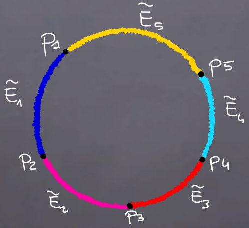
Ambrosio - Braides
Morel - Solimini
Brakke
Amato - Bellettini - Paolini
Carioni - Pluda

Equivalence of Steiner and Minimal Partitions Problem

Let Ω be convex.
There exists a minimizer to the Minimal partitions problem.

Let $\mathcal{C} = \{p_1, \dots, p_n\}$ be a finite collection of points on the boundary of $\Omega \subset \mathbb{R}^2$ convex.
Then the Steiner Problem and the Minimal partitions problem are equivalent.



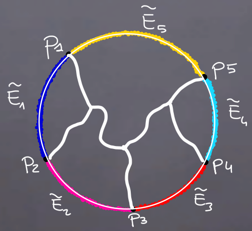


Ambrosio - Braides
Morel - Solimini
Brakke
Amato - Bellettini - Paolini
Carioni - Pluda

Equivalence of Steiner and Minimal Partitions Problem

Let Ω be convex.
There exists a minimizer to the Minimal partitions problem.

Let $\mathcal{C} = \{p_1, \dots, p_n\}$ be a finite collection of points on the boundary of $\Omega \subset \mathbb{R}^2$ convex.
Then the Steiner Problem and the Minimal partitions problem are equivalent.



Ambrosio - Braides
Morel - Solimini
Brakke
Amato - Bellettini - Paolini
Carioni - Pluda

Equivalence of Steiner and Minimal Partitions Problem

Let Ω be convex.
There exists a minimizer to the Minimal partitions problem.

Let $\mathcal{C} = \{p_1, \dots, p_n\}$ be a finite collection of points on the boundary of $\Omega \subset \mathbb{R}^2$ convex.
Then the Steiner Problem and the Minimal partitions problem are equivalent.

Ambrosio - Braides
Morel - Solimini
Brakke
Amato - Bellettini - Paolini
Carioni - Pluda

Paired Calibrations

Let $\Omega \subseteq \mathbb{R}^2$ be an open set.
A paired calibration for a Caccioppoli partition $\mathbf{E} = (E_1, \dots, E_n)$ is a collection of n approximately regular vector fields $\Phi_1, \dots, \Phi_n : \overline{\Omega} \to \mathbb{R}^2$ such that

1) $\operatorname{div} \Phi_i = 0$
2) $|\Phi_i - \Phi_j| \leq 1$ $\mathcal{H}^1$-a.e. in Ω
3) $(\Phi_i - \Phi_j) \cdot \nu_{ij} = 1$ $\mathcal{H}^1$ a.e. in Σ_{ij}

Lawlor - Morgan

Paired Calibrations

Let $\Omega \subseteq \mathbb{R}^2$ be an open set.
A paired calibration for a Caccioppoli partition $\mathbf{E} = (E_1, \dots, E_n)$ is a collection of n approximately regular vector fields $\Phi_1, \dots, \Phi_n : \overline{\Omega} \to \mathbb{R}^2$ such that

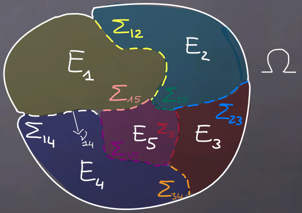
1) $\operatorname{div} \Phi_i = 0$
2) $|\Phi_i - \Phi_j| \leq 1$ $\mathcal{H}^1$-a.e. in Ω
3) $(\Phi_i - \Phi_j) \cdot \nu_{ij} = 1$ $\mathcal{H}^1$ a.e. in Σ_{ij}

If $\Phi = (\Phi_1, \dots, \Phi_n)$ is a paired calibration for $\mathbf{E} = (E_1, \dots, E_n)$ then $\mathbf{E} = (E_1, \dots, E_n)$ is a minimizer for the minimal partitions problem

Lawlor-Morgan

Calibration implies minimality

$$
\begin{aligned}
\mathcal{P}(\tilde{E}) &= \frac{1}{2}\sum_{i=1}^{m} P(\tilde{E}_i,\Omega) = \sum_{ij} \mathcal{H}^1(\tilde{\Sigma}_{ij}\cap\Omega) + \mathcal{H}^1(\Sigma_{m1}\cap\Omega)\\
&= \sum_{ij}\int_{\tilde{\Sigma}_{ij}\cap\Omega} 1 \, d\mathcal{H}^1 \overset{3}{=} \sum_{ij}\int_{\tilde{\Sigma}_{ij}\cap\Omega} (\Phi_i-\Phi_j)\cdot\nu_{ij}\, d\mathcal{H}^1\\
&= \sum_{i=1}^{m}\int_{\Omega} \Phi_i\cdot D\chi_{\tilde{E}_i} \overset{1}{=} \sum_{i=1}^{m}\int_{\Omega} \Phi_i\cdot D\chi_{E_i}\\
&= \sum_{ij}\int_{\Sigma_{ij}\cap\Omega} (\Phi_i-\Phi_j)\cdot\nu_{ij}\, d\mathcal{H}^1\\
&\overset{2}{\le} \sum_{ij}\int_{\Sigma_{ij}\cap\Omega} |\Phi_i-\Phi_j|\, d\mathcal{H}^1 \le \sum_{ij}\mathcal{H}^1(\Sigma_{ij}\cap\Omega)\\
&= \frac{1}{2}\sum_{i=1}^{m} P(E_i,\Omega) = \mathcal{P}(E)
\end{aligned}
$$



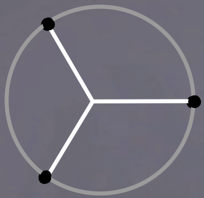
Looking at the differences

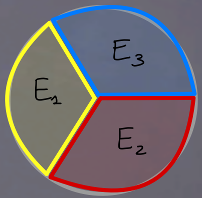
Given a Caccioppoli partition $E=(E_1,E_2,E_3)$ and the vector fields $\Psi_{12},\Psi_{23},\Psi_{31}:\overline{\Omega}\to\mathbb{R}$ such that $\operatorname{div}\Psi_{ij}=0$, $|\Psi_{ij}|\leq 1$, $\Psi_{ij}\cdot\nu_{ij}=1$ $\mathcal{H}^1$ a.e. in Σ_{ij} and with the property that $\Psi_{12}+\Psi_{23}+\Psi_{31}$ the collection of vector fields defined by $\Phi_1=(0,0)$, $\Phi_2=-\Psi_{12}$, $\Phi_3=\Psi_{31}$ is a calibration for $E=(E_1,E_2,E_3)$

Example : calibration of the triple junction

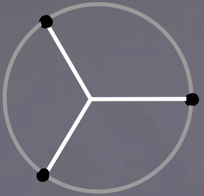
Lawlor - Morgan

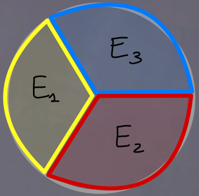
Example : calibration of the triple junction

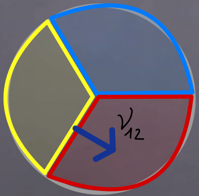


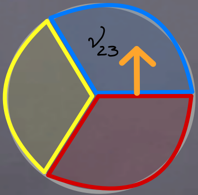


Example : calibration of the triple junction

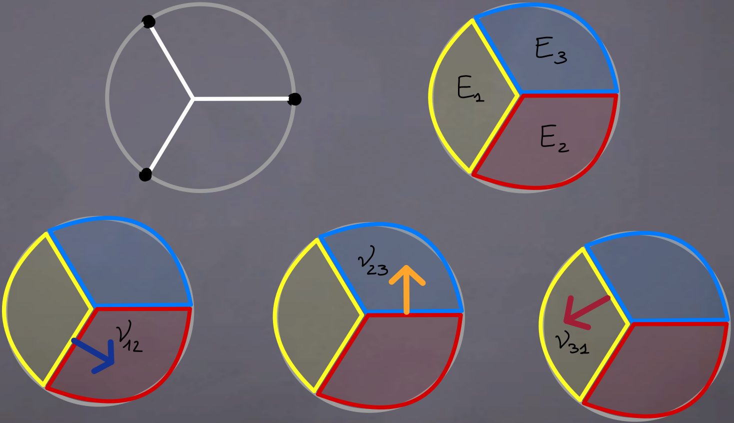








Example : calibration of the triple junction



$\Psi_{12} = \nu_{12}$ $\Psi_{23} = \nu_{23}$ $\Psi_{31} = \nu_{31}$

Example of non-existence of calibration

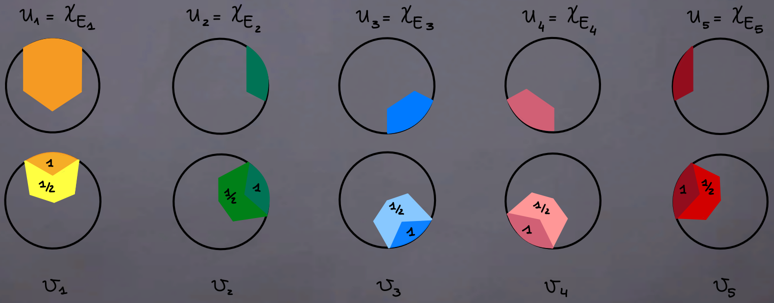
"Caccioppoli partition $\mathbf{E} = (E_1, \dots, E_m)$" is equivalent to
"collection of m functions $u_i \in BV(\Omega, \{0,1\})$ with $u_i = \chi_{E_i}$"

If $\Phi = (\Phi_1, \dots, \Phi_m)$ is a paired calibration for $u_1, \dots, u_m \in BV(\Omega, \{0,1\})$
then the collection $u_1, \dots, u_m$ minimize $\sum_{i=1}^{m} |Du_i|(\Omega)$
among all collections of m functions $v_i \in BV(\Omega, [0,1])$
with a finite number of values $\alpha_1, \dots, \alpha_k$, such that
$\sum_{i=1}^{m} v_i(x) = 1$ for a.e. $x \in \Omega$ and $\mathrm{tr}_{\Omega} u_i = \mathrm{tr}_{\Omega} v_i$

Bonafini-Orlandi-Oudet
Carioni-Pluda

Example of non-existence of calibration

"Caccioppoli partition $\mathbf{E} = (E_1, \dots, E_m)$" is equivalent to "collection of m functions $u_i \in BV(\Omega, \{0,1\})$ with $u_i = \chi_{E_i}$"



A local minimality result

Let $\mathcal{N}$ be a minimal network contained in D homeomorphic to a closed disk and with endpoints $p_1, \dots, p_m \in \partial D$

Then there exists a bounded open set Ω and $E = (E_1, E_2, E_3)$ a Caccioppoli partition of Ω such that $\mathcal{N} = \Omega \cap \bigcup_{i=1}^{3} \partial^* E_i$

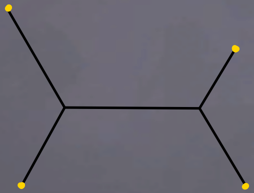
Moreover there exists a paired calibration for in Ω

In particular E is a minimizer of among all having the same trace of E on $\partial\Omega$

Pluda - Pozzetta

Proof

Prototypical configurations :

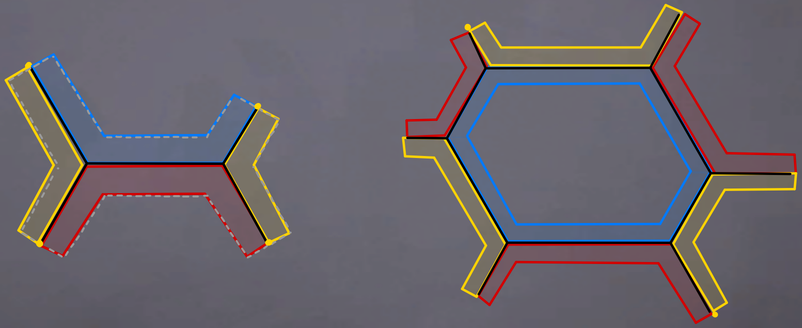


Proof

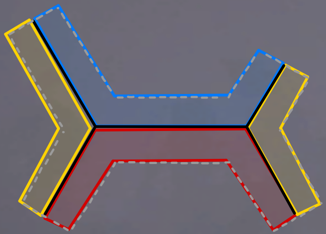
We take a truncated δ-neighborhood

Proof

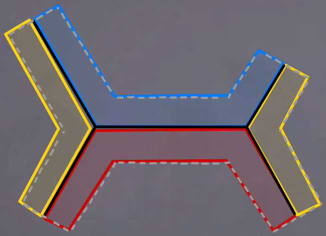
We define the partition

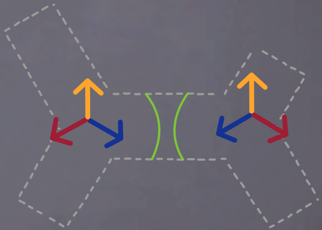


Proof

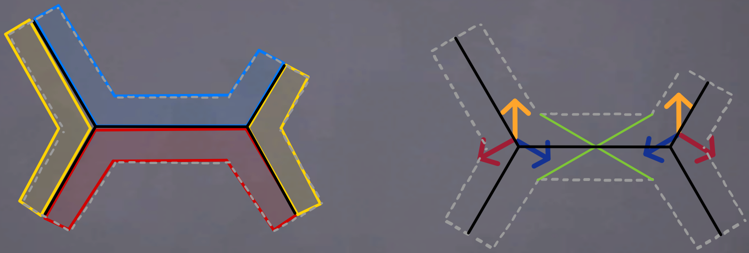


Proof

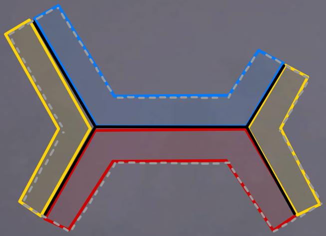


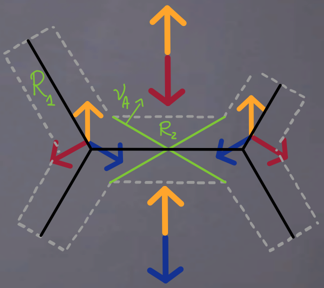


Proof



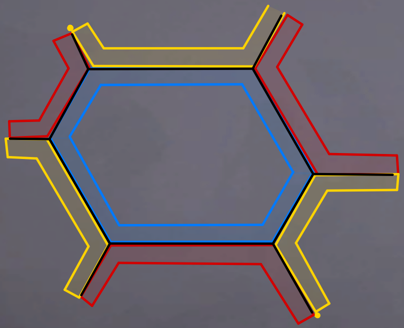
Proof

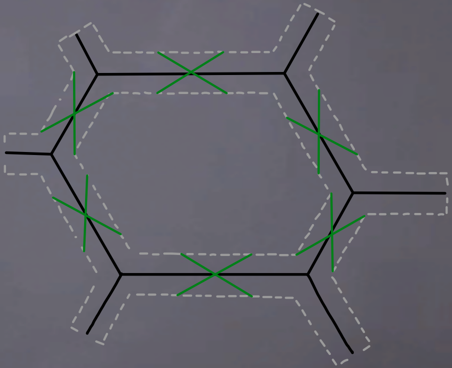




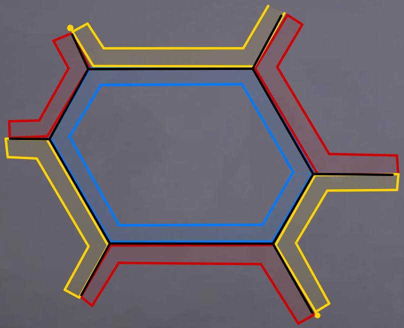
$$\mathrm{tr}_{R_1}\left(\Psi_{ij}|_{R_1}\right)\cdot \nu_A = -\mathrm{tr}_{R_2}\left(\Psi_{ij}|_{R_2}\right)\cdot \nu_A$$

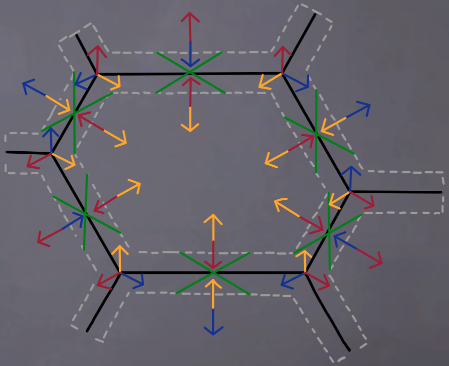
Proof





Proof





Currents with coefficients in $\mathbb{R}^m$

A 1-form ω with values in $\mathbb{R}^m$ is an array $\omega = (\omega_1, \dots, \omega_m)$ of 1-forms

$C_c^\infty(\mathbb{R}^2, \Lambda^1_m(\mathbb{R}^2))$ is the space of 1-forms with values in $\mathbb{R}^m$

A 1-current with coefficients in $\mathbb{R}^m$ is a linear and continuous map $T : C_c^\infty(\mathbb{R}^2, \Lambda^1_m(\mathbb{R}^2)) \to \mathbb{R}$

Its mass is defined as

$$\mathbb{M}(T) := \sup\{T(\omega) : \omega \in C_c^\infty(\mathbb{R}^2, \Lambda^1_m(\mathbb{R}^2)) \text{ with } \|\omega\|_{com} \leq 1\}$$

and its boundary is the 0-current defined by

$$\partial T(\omega) = T(d\omega) \qquad \forall \omega \in C_c^\infty(\mathbb{R}^2, \Lambda^1_m(\mathbb{R}^2))$$

Fleming, White

A mass-minimization problem

Let $\{g_1, \ldots, g_{m-1}\}$ be the canonical base of $\mathbb{R}^{m-1}$ and $g_m := -\sum_{i=1} g_i$

We define a norm $\|\cdot\|$ on $\mathbb{R}^{m-1}$ in such a way that given I any subset of $\{1, \ldots, m-1\}$ it holds $\|\sum_{i\in I} g_i\| = 1$

Given $\{p_1, \ldots, p_n\}$ a finite collection of points in $\mathbb{R}^2$ we define the 0-current $B = g_1\delta_{p_1} + \ldots + g_m\delta_{p_m}$

Problem $(*)$:

$\inf\{\mathbb{M}(T) : T$ 1-rectifiable current with coefficients in $\mathbb{Z}^{m-1}$, $\partial T = B\}$.

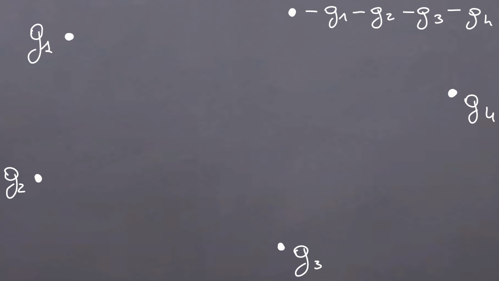
A mass-minimization problem

Let $\{g_1, \dots, g_{m-1}\}$ be the canonical base of $\mathbb{R}^{m-1}$ and $g_m := -\sum_{i=1} g_i$

given I any subset of $\{1, \dots, m-1\}$ it holds $\|\sum_{i\in I} g_i\| = 1$

$B = g_1 \delta_{P_1} + \dots + g_m \delta_{P_m}$

$\inf\{\mathbb{M}(T) : T$ 1-rectifiable current with coefficients in $\mathbb{Z}^{m-1}$; $\partial T = B\}$.



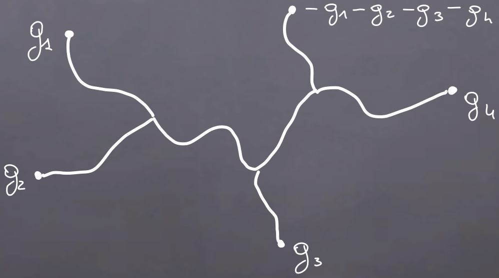
A mass-minimization problem

Let $\{g_1, \dots, g_{m-1}\}$ be the canonical base of $\mathbb{R}^{m-1}$ and $g_m := -\sum_{i=1} g_i$

given I any subset of $\{1, \dots, m-1\}$ it holds $\|\sum_{i\in I} g_i\| = 1$

$B = g_1\delta_{P_1} + \dots + g_m\delta_{P_m}$

$\inf\{ \mathbb{M}(T) : T$ 1-rectifiable current with coefficients in $\mathbb{Z}^{m-1}$; $\partial T = B\}$.



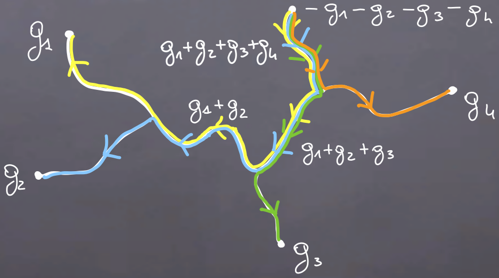
A mass-minimization problem

Let $\{g_1,\dots,g_{m-1}\}$ be the canonical base of $\mathbb{R}^{m-1}$ and $g_m := -\sum_{i=1} g_i$

given I any subset of $\{1,\dots,m-1\}$ it holds $\|\sum_{i\in I} g_i\| = 1$

$B = g_1\delta_{P_1} + \dots + g_m\delta_{P_m}$

$\inf\{\mathbb{M}(T) : T$ 1-rectifiable current with coefficients in $\mathbb{Z}^{m-1}$; $\partial T = B\}$.



Equivalence of Steiner and the mass minimization Problem

There exists a minimizer to the mass-minimization problem (*)

Marchese - Massaccesi

Equivalence of Steiner and the mass minimization Problem

There exists a minimizer to the mass-minimization problem (*)

The Steiner Problem and the mass-minimization problem (*) are equivalent.

Marchese - Massaccesi

Calibrations for $\mathcal{G}$-currents

Let $\mathcal{G}$ be a normed subgroup of $\mathbb{R}^{m-1}$

Let $T = [\Sigma, \tau, \theta]$ be a 1-rectifiable current with coefficients in $\mathcal{G}$ and $\omega \in C_c^\infty(\mathbb{R}^2, \Lambda_m^1(\mathbb{R}^2))$

Then ω is a calibration for T if

1) $d\omega = 0$

2) $\|\omega\|_{com} \leq 1$

3) $\langle \omega(x)\tau(x), \theta(x) \rangle = \|\theta(x)\|$ for $\mathcal{H}^1$-a.e. $x \in \Sigma$

Calibrations for $\mathcal{G}$-currents

Let $\mathcal{G}$ be a normed subgroup of $\mathbb{R}^{m-1}$

Let $T = [\Sigma, \tau, \theta]$ be a 1-rectifiable current with coefficients in $\mathcal{G}$ and $\omega \in C^\infty_c(\mathbb{R}^2; \Lambda^1_m(\mathbb{R}^2))$

Then ω is a calibration for T if

1) $d\omega = 0$

2) $\|\omega\|_{com} \leq 1$

3) $\langle \omega(x)\tau(x), \theta(x) \rangle = \|\theta(x)\|$ for $\mathcal{H}^1$-a.e. $x \in \Sigma$

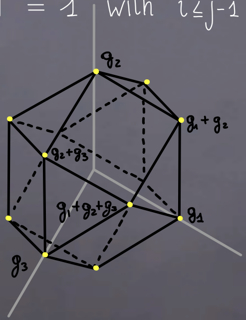
If ω is a calibration for T then T is mass minimizing in its homology class

Equivalence of calibrations

Let $\{g_1, \dots, g_{m-1}\}$ be the canonical base of $\mathbb{R}^{m-1}$ and $g_m := -\sum_{i=1} g_i$

We define a norm $\|\cdot\|$ on $\mathbb{R}^{m-1}$ characterized by the property that its unit ball is the smallest such that

$$\left\|\sum_{k=i}^{j-1} g_k\right\| = 1 \quad \text{with } i \le j-1 \qquad i,j \in \{1, \dots, m\}$$



Carioni - Pluda

Equivalence of calibrations

Let $\{g_1,\dots,g_{m-1}\}$ be the canonical base of $\mathbb{R}^{m-1}$ and $g_m := -\sum_{i=1} g_i$

We define a norm $\|\cdot\|$ on $\mathbb{R}^{m-1}$ characterized by the property that its unit ball is the smallest such that

$$\left\|\sum_{k=i}^{j-1} g_k\right\| = 1 \quad \text{with } i \leq j-1 \qquad i,j \in \{1,\dots,m\}$$

Then there exists a permutation of the labelling of the points $p_1,\dots,p_m$ such that gives the equivalence of the paired calibrations and the calibrations for G-currents

Carioni - Pluda

From minimal networks to $\mathcal{G}$-currents

Let $g_1 = (1,0)$, $g_2 = \left(-\frac{1}{2}, -\frac{\sqrt{3}}{2}\right)$ and $g_3 = -g_1 - g_2$.

We choose a norm $\|\cdot\|$ on $\mathbb{R}^2$ such that

$$\|g_1\| = \|g_2\| = \|g_1 + g_2\| = 1$$

Let $\hat{\mathcal{G}}$ be the discrete group generated by g_1 and g_2 w.r.t. addition

Pluda - Pozzetta

From minimal networks to $\mathcal{G}$-currents

Let $g_1 = (1,0)$, $g_2 = (-\frac{1}{2}, -\frac{\sqrt{3}}{2})$ and $g_3 = -g_1 - g_2$.

We choose a norm $\|\cdot\|$ on $\mathbb{R}^2$ such that

$$\|g_1\| = \|g_2\| = \|g_1 + g_2\| = 1$$

Let $\widehat{\mathcal{G}}$ be the discrete group generated by g_1 and g_2 w.r.t. addition

There is a canonical way to associate to $\widehat{\mathcal{N}}$ minimal network a 1-rectifiable current $\widehat{T} = [\widehat{\Sigma}, \widehat{\tau}, \widehat{\theta}]$ with coefficients in $\widehat{\mathcal{G}}$ such that

$$\operatorname{supp}(\widehat{T}) = \widehat{\mathcal{N}} \qquad L(\widehat{\mathcal{N}}) = \mathbb{M}(\widehat{T})$$

and $\widehat{B} = \partial \widehat{T} = c_1 \delta_{p_1} + \ldots + c_m \delta_{p_m}$ with $c_i \in \{\pm g_1, \pm g_2, \pm g_3\}$

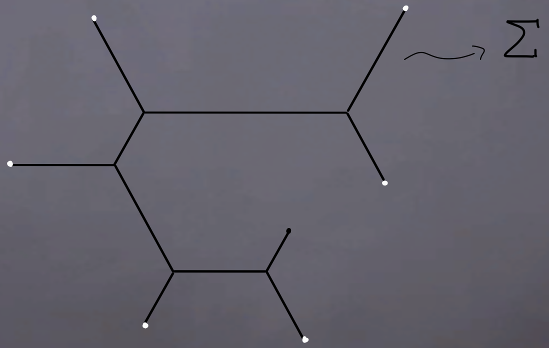
Pluda-Pozzetta

From minimal networks to $\mathcal{G}$-currents

There is a canonical way to associate to $\widehat{\mathcal{N}}$ minimal network a 1-rectifiable current $\widehat{T} = [\widehat{\Sigma}, \widehat{\tau}, \widehat{\theta}]$ with coefficients in $\widehat{\mathcal{G}}$

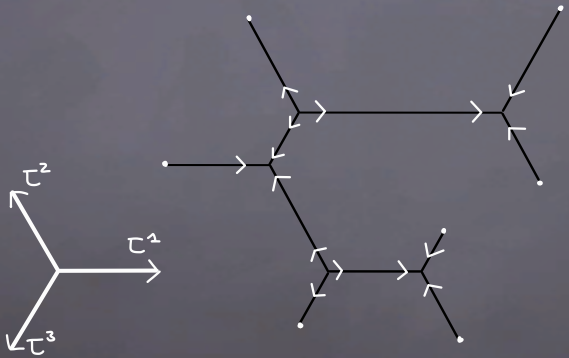
From minimal networks to $\mathcal{G}$-currents

There is a canonical way to associate to $\hat{\mathcal{N}}$ minimal network a 1-rectifiable current $\hat{T} = [\hat{\Sigma}, \hat{\tau}, \hat{\theta}]$ with coefficients in $\hat{\mathcal{G}}$



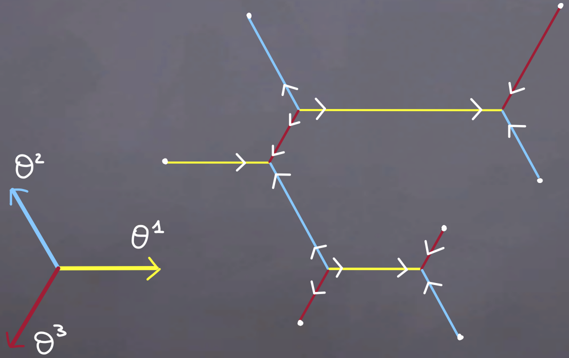
From minimal networks to $\mathcal{G}$-currents

There is a canonical way to associate to $\hat{\mathcal{N}}$ minimal network a 1-rectifiable current $\hat{T} = [\hat{\Sigma}, \hat{\tau}, \hat{\theta}]$ with coefficients in $\hat{\mathcal{G}}$



From minimal networks to $\mathcal{G}$-currents

There is a canonical way to associate to $\widehat{\mathcal{N}}$ minimal network a 1-rectifiable current $\widehat{T} = [\widehat{\Sigma}, \widehat{\tau}, \widehat{\theta}]$ with coefficients in $\widehat{\mathcal{G}}$



A global minimality result

There exists ω calibration for the 1-rectifiable current $\hat{T}$ with coefficients in $\hat{\mathcal{G}}$ canonically associated to $\hat{\mathcal{N}}$ minimal network

In particular $\hat{T}$ is mass minimizing among 1-normal currents with coefficients in $\mathbb{R}^2$ with $\mathcal{C}$ and with $\hat{\mathcal{C}}$ boundary $\hat{B}$

Pluda - Pozzetta

A global minimality result

There exists ω calibration for the 1-rectifiable current $\widehat{T}$ with coefficients in $\widehat{\mathcal{G}}$ canonically associated to $\widehat{\mathcal{N}}$ minimal network
In particular $\widehat{T}$ is mass minimizing among 1-normal currents with coefficients in $\mathbb{R}^2$ with and with boundary $\widehat{B}$

Take $\omega = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

Pluda-Pozzetta

Łojasiewicz-Simon inequality

Let $\mathcal{N}_* = (G\,(\gamma^1_*, \dots, \gamma^N_*))$ be a minimal network.
Then there exists $c > 0$, $\varepsilon > 0$, $\theta \in (0, \tfrac{1}{2}]$ such that the following holds

if $\mathcal{N} = (G, (\gamma^1_0, \dots, \gamma^N_0))$ is a regular network such that

$$\sum \| \gamma^i - \gamma^i_* \|_{H_2} < \varepsilon$$

Then

$$| L(\mathcal{N}) - L(\mathcal{N}_*) |^{1-\theta} \leq c \left(\sum_i \int_0^1 | \vec{k} |^2 \, ds \right)^{1/2} = c \, \| \vec{k}_{\mathcal{N}} \|_2$$